HOMEWORK 3 SOLUTIONS

(2.2.16) Evaluate $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$.

Solution. As $h \to 0$, both the numerator and the denominator approach 0, so I know I have to do a little work to evaluate this limit. This type of problem usually requires me to multiply by the conjugate, so I'll do that.

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right) &= \lim_{h\to 0}\frac{1+h-1}{h(\sqrt{1+h}+1)}\\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)} = \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.\end{aligned}$$

□

(2.2.20) Evaluate $\lim_{h\to 0} \dfrac{(3+h)^{-1}-3^{-1}}{h}$.

Solution. The laws of exponents mean that they want me to evaluate

$$\lim_{h\to 0}\frac{\left(\frac{1}{3+h}-\frac{1}{3}\right)}{h}.$$

I know when I come across a problem like this that I'm supposed to express the difference of fractions as a single fraction by finding a common denominator. I'll do that and then proceed.

$$\lim_{h\to 0}\frac{\left(\frac{1}{3+h}-\frac{1}{3}\right)}{h} = \lim_{h\to 0}\frac{\left(\frac{3-(3+h)}{3(3+h)}\right)}{h} = \lim_{h\to 0}\frac{3-(3+h)}{3h(3+h)} = \lim_{h\to 0}\frac{-h}{3h(3+h)} = \lim_{h\to 0}\frac{-1}{3(3+h)} = -\frac{1}{9}.$$

□

(2.4.34) Suppose that a function f is continuous on $[0,1]$ except at 0.25 and that $f(0)=1$ and $f(1)=3$. Let $N=2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the intermediate value theorem and one showing that f might still satisfy the conclusion of the intermediate value theorem (even though it doesn't satisfy the hypothesis).

Solution. There are lots of possible solutions, but here is an answer that fits the bill. □

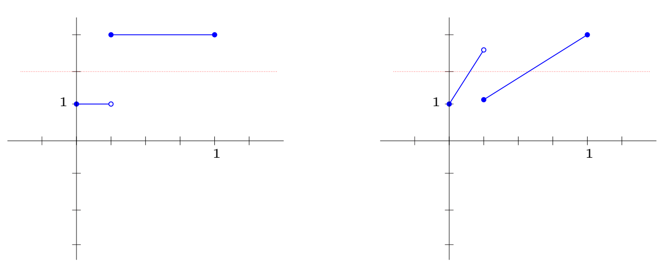


FIGURE 1. The first does not satisfy IVT for $N=2$, the second does

(2.4.35) Let $f(x) = x^2 + 10\sin(x)$, show that there is a number c such that $f(c) = 1000$.

Solution. This sounds a lot like the intermediate value theorem. In order to show that there is a number c so that $f(c) = 1000$, I need to find points a and b with $f(a) < 1000$ and $f(b) > 1000$, verify that $f(x)$ is continuous on the closed interval between a and b, and apply the intermediate value theorem.

For this, first I notice that $f(0) = 0^2 + 10\sin(0)$. Now I know that $\sin(0) = 0$ because I'm a geek, and so I get $f(0) = 0+0 = 0 < 1000$. If you don't remember that $\sin(0) = 0$, though, you can reason as follows. I know $\sin(0)$ is some number between -1 and 1 (since the sine of any number is between -1 and 1), and therefore $10\sin(0)$ is somewhere between -10 and 10. But then $0 + 10\sin(0) < 1000$ (since 0 plus a number somewhere between -10 and 10 isn't very big...at most, it's 10). Either way you slice it, you conclude

$$f(0) < 1000.$$

Now I need to find a point where $f(b) > 1000$. Now I know that $f(100) = 100^2 + 10\sin(100) = 10000 + 10\sin(100)$. Even though I don't know what $\sin(100)$ is, I know it is something between -1 and 1, and so $10(\sin(100))$ is somewhere between -10 and 10. Hence $100^2 + 10\sin(100)$ is somewhere between $10000 - 10$ and $10000 + 10$. In any case, I know that

$$f(100) > 1000.$$

Now since $f(x)$ is continuous on $[0, 100]$ (it is the sum of functions which are continuous everywhere, so it is itself continuous everywhere), and since $f(0) < 1000$ and $f(100) > 1000$, the intermediate value theorem tells us there exists some number c in $(0, 100)$ so that $f(c) = 1000$. We win! □

(2.4.47) Is there a number that is exactly 1 more than its cube?

Solution. We begin by translating this statement into an equation that we can use the intermediate value theorem to solve. To say that a number is 1 more than its cube is to say that the number is a solution to the equation

$$x = x^3 + 1.$$

Since the intermediate value theorem lets us conclude when a given function achieves a certain numerical value, we re-translate this equation into the form $f(x) = 0$ for some $f(x)$. In this case, a solution to the equation $x = x^3 + 1$ is the same as a solution to the equation

$$x^3 - x + 1 = 0$$

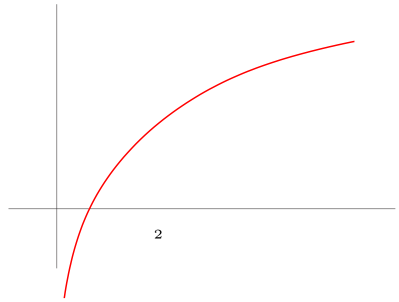
(we've just moved the x onto the other side of the equation). This is great, because we can use the intermediate value theorem to find a solution to this equation: all we have to do is verify the function on the left-hand side is continuous and that it takes on a value less than 0 and a value greater than 0.

We begin by noting that $x^3 - x + 1$ is a polynomial, and hence is continuous everywhere. Now we're on the hunt for numbers a and b with $f(a) < 0$ and $f(b) > 0$. We'll try some values that are particularly easy to evaluate:

$$\begin{aligned} f(0) &= 0 - 0 + 1 = 1 > 0 \\ f(1) &= 1 - 1 + 1 = 1 > 0 \\ f(2) &= 8 - 2 + 1 = 7 > 0 \\ f(3) &= 27 - 3 + 1 = 25 > 0 \\ f(-1) &= -1 + 1 + 1 = 1 > 0 \\ f(-2) &= -8 + 2 + 1 = -5 < 0. \end{aligned}$$

Great! Now we can use the intermediate value theorem. Since $x^3 - x - 1$ is continuous on $[-2, -1]$, and $f(-2) < 0$ and $f(-1) > 0$, the intermediate value theorem tells us there exists a number c in $(-2, -1)$ which is a solution to $x^3 - x + 1 = 0$. This is exactly what we want. □

(2.7.2) For the function f whose graph is shown, arrange the following numbers in increasing order and explain your reasoning: $0 \quad f'(2) \quad f(3) - f(2) \quad \frac{1}{2}\left[f(4) - f(2)\right]$.



Solution. This one is a little tricky since we're not given an algebraic expression for the function. So how do we proceed? We have a pretty good handle on 0 and $f'(2)$ (which is the slope of the tangent line at 2), but what about the other two?

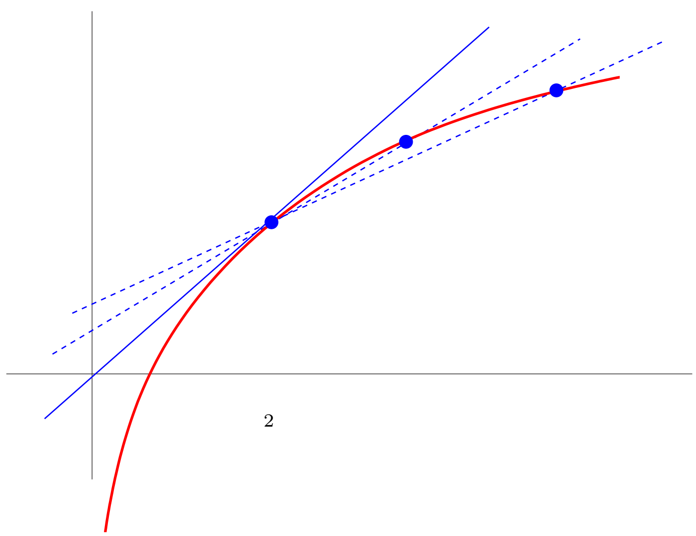
The key is to notice that these numbers also represent slopes. For instance, I can rewrite

$$\frac{1}{2}\left[f(4) - f(2)\right] = \frac{f(4) - f(2)}{2} = \frac{f(4) - f(2)}{4 - 2}.$$

The quantity on the right is exactly the slope of the (secant) line through $(2, f(2))$ and $(4, f(4))$. Similarly

$$f(3) - f(2) = \frac{f(3) - f(2)}{1} = \frac{f(3) - f(2)}{3 - 2},$$

and the quantity on the right is the slope of the (secant) line through $(2, f(2))$ and $(3, f(3))$. With these things in mind, I can draw in the appropriate lines and then compare slopes.



All lines have positive slope, so 0 is the smallest number in my list. Now the tangent line is the steepest of the lines, so it has the largest slope. The secant line through $(2, f(2))$ and $(4, f(4))$ is the least steep, so it has the smallest slope. Hence we find

$$0 < \frac{1}{2}\left[f(4) - f(2)\right] < f(3) - f(2) < f'(2).$$

□

(2.7.5) Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

Solution. To solve a problem like this, I usually graph a set of axes and start drawing little pieces of my funtion. For instance, I know that my function should pass through 0 at 0 (since $f(0) = 0$), so I'll put a dot at the point $(0, 0)$ so I know my function has to pass through there. To remind myself that $f'(0) = 3$, I usually draw a little shred of a line of slope 3 through the point $(0, 0)$. This way I will know the function has to be tangent to this line at 0.

Now I'm not told what $f(1)$ is supposed to be, but I know that $f'(1) = 0$. So I'll just plop a point onto my graph at $x = 1$, say the point $(1, 2)$. At that point I'll draw a little shred of a line of slope zero, again to remind me that the function has to be tangent to this line at 1.

Again, I'm not told what $f(2)$ is, but I'm forced into having $f'(2) = -1$. I'll plop a point down randomly for $x = 2$, say $(2, -2)$, and then draw a shred of a line of slope -1 through this point.

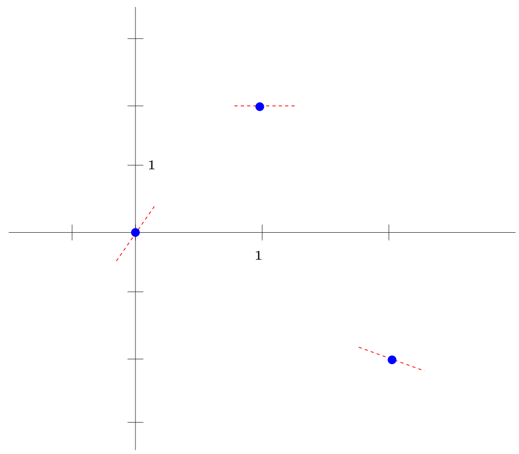


FIGURE 2. Graphing the information f is supposed to satisfy

With all these little bits taken care of, I can now fill in the rest of my curve. There are lots of ways to do this, but here's one possibility.

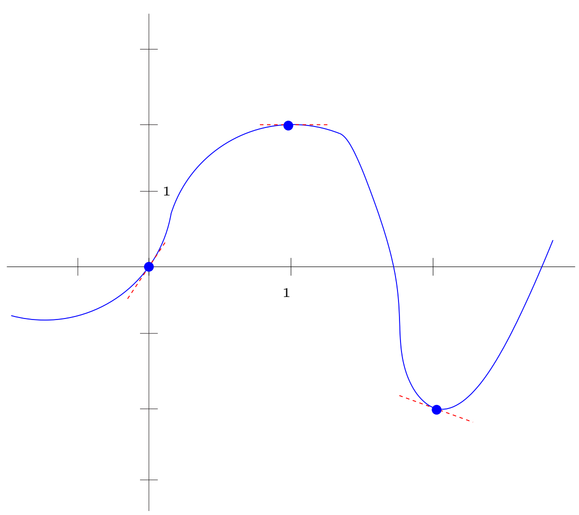


FIGURE 3. One possibility for filling out the function

□

(2.7.8) If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.

Solution. I know that $g'(0)$ is defined to be a certain limit, so I'll compute $g'(0)$ in this way:

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{1 - h^3 - 1}{h} = \lim_{h\to 0} \frac{-h^3}{h} = \lim_{h\to 0} -h^2 = 0.$$

With this information, I can now write the equation of the line tangent to $g(x)$ at $(0, 1)$. Since this line has slope 0 (we just computed this) and passes through $(0, 1)$, the equation of the line is

$$y - 1 = 0(x - 0) \text{ or equivalently} y = 1.$$

Notice that if I graph $y = g(x)$ I can see that the slope of the tangent line at 0 is 0 (trying graphing this function to see why this is true). In general you won't be able to 'eyeball' $f'(a)$ for random f and a, but in the special cases where $f'(a) = 0$ you will be able to notice this visually. □

(2.7.9(a)) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.

Solution. I know that $F'(2)$ is defined to be a certain limit, so I'll compute $F'(2)$ by evaluating

$$\lim_{x\to 2} \frac{F(x) - F(2)}{x - 2} = \lim_{x\to 2} \frac{\left(\frac{5x}{1+x^2} - \frac{10}{5}\right)}{x - 2} = \lim_{x\to 2} \frac{\left(\frac{5x}{1+x^2} - 2\right)}{x - 2}$$

$$= \lim_{x\to 2} \frac{\left(\frac{5x - 2(1+x^2)}{1+x^2}\right)}{x - 2} = \lim_{x\to 2} \frac{5x - 2(1+x^2)}{(x-2)(1+x^2)}$$

$$= \lim_{x\to 2} \frac{5x - 2 - 2x^2}{(x-2)(1+x^2)} = \lim_{x\to 2} \frac{(-2x+1)(x-2)}{(x-2)(1+x^2)} = \lim_{x\to 2} \frac{(-2x+1)}{(1+x^2)} = \frac{-4+1}{1+4} = -\frac{3}{5}.$$

□

(2.7.16) Find $f'(a)$ for $f(x) = \dfrac{x^2+1}{x-2}$ and $a = 1$.

Solution. We know that $f'(a)$ is given by a certain limit, so we'll compute that limit:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\left(\frac{(1+h)^2+1}{(1+h)-2} - (-2)\right)}{h} = \lim_{h\to 0} \frac{\left(\frac{(1+h)^2+1}{h-1} + 2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{(1+h)^2+1+2(h-1)}{h-1}\right)}{h} = \lim_{h\to 0} \frac{(1+h)^2+1+2(h-1)}{h(h-1)} = \lim_{h\to 0} \frac{(1+2h+h^2+1+2h-2)}{h(h-1)}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h(h-1)} = \lim_{h\to 0} \frac{h(4+h)}{h(h-1)} = \lim_{h\to 0} \frac{4+h}{(h-1)} = -\frac{4}{1} = -\frac{4}{1} = -4$$

□

(2.7.17) Find $f'(a)$ for $f(x) = \dfrac{1}{\sqrt{x+2}}$ and $a = 2$.

Solution. To compute $f'(a)$ I use the limit definition. I can choose to evaluate one of two seemingly different limits, so for this one I'll do the '$x \to a$' limit (instead of the '$h \to 0$' version).

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{\left(\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{2+2}}\right)}{x-2} = \lim_{x\to 2} \frac{\left(\frac{1}{\sqrt{x+2}} - \frac{1}{2}\right)}{x-2}$$

$$= \lim_{x\to 2} \frac{\left(\frac{2-\sqrt{x+2}}{2\sqrt{x+2}}\right)}{x-2} = \lim_{x\to 2} \frac{2-\sqrt{x+2}}{2(x-2)\sqrt{x+2}} = \lim_{x\to 2} \frac{2-\sqrt{x+2}}{2(x-2)\sqrt{x+2}} \cdot \left(\frac{2+\sqrt{x+2}}{2+\sqrt{x+2}}\right)$$

$$= \lim_{x\to 2} \frac{4-(x+2)}{2(x-2)\sqrt{x+2}(2+\sqrt{x+2})} = \lim_{x\to 2} \frac{2-x}{2(x-2)\sqrt{x+2}(2+\sqrt{x+2})} = \lim_{x\to 2} \frac{-(x-2)}{2(x-2)\sqrt{x+2}(2+\sqrt{x+2})}$$

$$= \lim_{x\to 2} \frac{-1}{2\sqrt{x+2}(2+\sqrt{x+2})} = \frac{-1}{2\sqrt{2+2}(2+\sqrt{2+2})} = -\frac{1}{2\cdot 2(2+2)} = -\frac{1}{16}$$

□

(2.7.20) The expression

$$\lim_{h\to 0} \frac{\sqrt[4]{16+h} - 2}{h}$$

represents $f'(a)$ for what function $f(x)$ and what value of a?

Solution. It seems they are using the definition

$$f'(a) = \lim_{h\to 0} f(a+h) - f(a)h$$

to compute the derivative. Hence the first term in the numerator shoul be $f(a+h)$. Hence it's reasonable to think that $a = 16$ and that $f(x) = \sqrt[4]{x}$. This is exactly the right answer (since $\sqrt[4]{16} = 2$). □

(2.7.22) The expression

$$\lim_{x\to \pi/4} \frac{\tan(x) - 1}{x - \pi/4}$$

represents $f'(a)$ for what function $f(x)$ and what value of a?

Solution. This time they are using the definition

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

to calculate the derivative. That means that I can read off a as the term in the limit (in this case, $a = \frac{\pi}{4}$) and I can read off $f(x)$ as the first term in the numerator. So I have $f(x) = \tan(x)$ and $a = \frac{\pi}{4}$. □