**HOMEWORK 4 SOLUTIONS**

Thanks to Ben Williams, CA extraordinaire, for writing up these solutions!

(2.8.2) We can estimate the derivatives by sketching the tangent line at each point and then approximating the slope of the tangent line. Doing this, we have

- $f'(0) \approx -3$ (any guess less than $-1.5$ is good),
- $f'(1) = 0$,
- $f'(2) \approx 2$,
- $f'(3) \approx 2$,
- $f'(4) = 0$, and
- $f'(5) \approx -1$.

Plotting these points and using them as a guide, the graph of the derivative looks something like this:

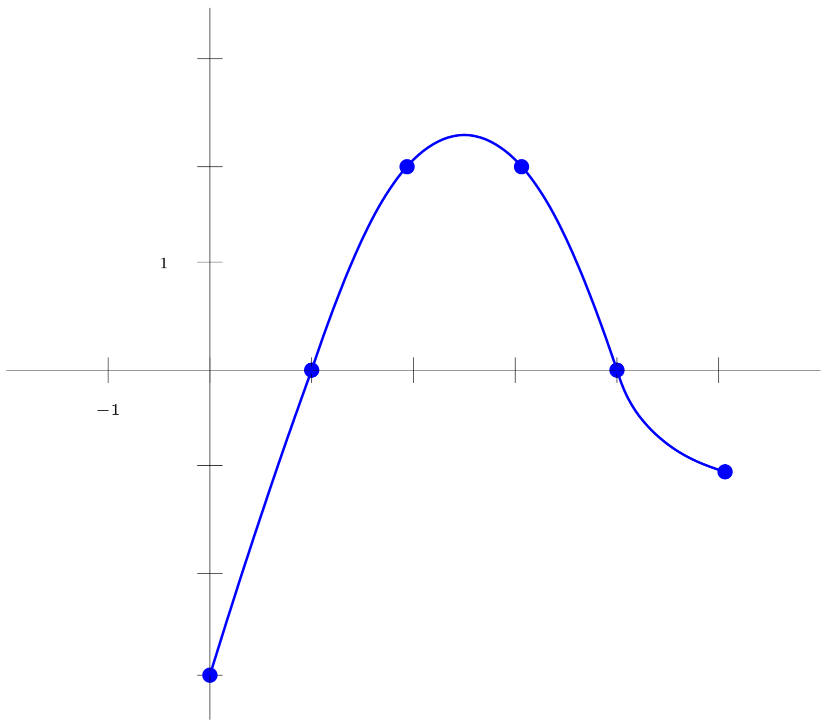


FIGURE 1. A sketch of $f'(x)$

(2.8.3) This problem can be solved by considering points where the derivative is 0,

- (a) has two points where the derivative is 0, so corresponds to II
- (b) has no points where the derivative is 0, and two points where the derivative is undefined (cusps), and so corresponds to IV.
- (c) has only one point where the derivative is 0, and therefore corresponds to I
- (d) has three points where the derivative is 0, and therefore corresponds to III.

(2.8.22) We evaluating the derivative by using the definition of the derivative as a limit. We'll multiply by the conjugate in order to get the simplification we want.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)+\sqrt{x+h}]-(x+\sqrt{x})}{h}$$

$$= \lim_{h\to 0} \frac{h+\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\left(\frac{h}{h}+\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$= \lim_{h\to 0}\left(1+\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = 1+\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= 1+\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = 1+\frac{1}{2\sqrt{x}}$$

(2.8.23) Again, we use the definition of the derivative as a limit to compute. We'll express the difference of fractions as a single fraction and pray for some cancellation.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\cdot\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}-\sqrt{1+2x}}$$

$$= \lim_{h\to 0}\frac{1+2(x+h)-(1+2x)}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x}} = \lim_{h\to 0}\frac{2h}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x}}$$

$$= \lim_{h\to 0}\frac{2}{(\sqrt{1+2(x+h)}+\sqrt{1+2x}} = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$$

(2.8.24) No surprise: we'll use the definition of the derivative as a limit to compute.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{3+(x+h)}{1-3(x+h)}-\frac{3+x}{1-3x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1-3x}{1-3x}\cdot\frac{3+(x+h)}{1-3(x+h)}-\frac{3+x}{1-3x}\cdot\frac{1-3(x+h)}{1-3(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{(3+x+h)(1-3x)-(1-3x-3h)(3+x)}{h(1-3x-3h)(1-3x)}$$

$$= \lim_{h\to 0}\frac{[3-9x+x-3x^2+h-3xh]-[3+x-9x-3x^2-9h-3xh]}{h(1-3x-3h)(1-3x)}$$

$$= \lim_{h\to 0}\frac{10h}{h(1-3x-3h)(1-3x)} = \lim_{h\to 0}\frac{10}{(1-3x-3h)(1-3x)} = \frac{10}{(1-3x)^2}$$

(2.8.31) $f$ is not differentiable at $-4$, where there is a cusp (or corner), or at 0, where there is a discontinuity.

(2.8.32) $f$ is not differentiable at 0, where there is a discontinuity, or at 3, where there is a cusp (or corner).

(2.8.33) $f$ is not differentiable at $-1$, where there is a vertical tangent, or at 4, where there is a cusp (or corner).

(2.8.34) $f$ is not differentiable at $-1$, where there is a discontinuity, or at 2, where there is a cusp (or corner).

(2.8.37) Note that each of $a, b, c$ cross the $x$-axis only once. Since $c$ plainly has two points where its derivative is 0, it follows that neither $a$ nor $b$ is the derivative of $c$. So $c = f''$. Now, at the point at which $c$ is 0, $a$ is increasing and has slope greater than 0, so $c$ is not the derivative of $a$. So we must have $b = f'$, and $a = f$.

(2.8.38) We'll chart some basic information about each function.

| graph | points where function is 0 | points where derivative is 0 |
|---|---|---|
| $a$ | 1 | 1 |
| $b$ | 1 | 1 |
| $c$ | 2 | 1 |
| $d$ | 3 | 2 |

From this, we see that $d$ is not the derivative of any of $a, b, c$, since $d$ has more zeros than any of $a, b$, or $c$ has tangent lines with slope 0. Hence $d = f$. Since $c$ is the only function which has as many zeros as $d$ has tangent lines with slope 0, we see $c$ is the derivative of $d$, and hence $c = f'$.

Initially, $c$ is decreasing, so its derivative is initially negative. Since $a$ is always positive, this means that $a$ cannot be the derivative of $c$. Hence $b$ is the derivative of $c$, and so $b = f''$. This leaves $a = f'''$.