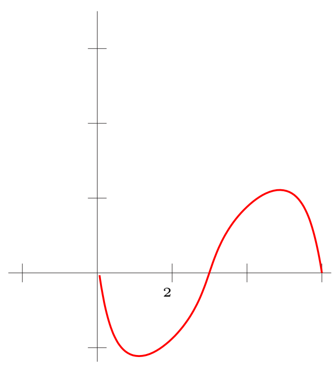
HOMEWORK 7 SOLUTIONS

(2.9.1) Given the graph of $f'(x)$,

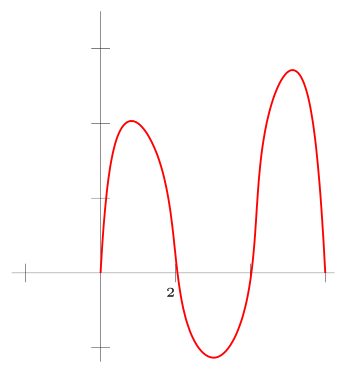
- On what intervals is f increasing or decreasing?
- At what Values of x does f have a local maximum or minimum.
- If it is known that $f(0) = 0$, sketch a possible graph of f.

Solution. Here we go.

- Since $f'(x) > 0$ on $(1, 5)$, our dictionary tells us $f(x)$ is increasing on this interval. Similarly, since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, the dictionary tells us $f(x)$ is decreasing on this interval.
- By the first derivative test, $f(x)$ has a local maximum at $x = 5$ and a local minimum at $x = 1$.
- Our analysis tells us a lot about the graph. We could also not that since $f'(x)$ is increasing on $(0, 3)$ and decreasing on $(3, 6)$, the graph of $f(x)$ is concave up on $(0, 3)$ and concave down on $(3, 6)$. With all this information, we can sketch the graph of $f(x)$.



A solution for 2.9.1



A solution for 2.9.2

□

(2.9.2) Given the graph of $f'(x)$,

- On what intervals is f increasing or decreasing?
- At what Values of x does f have a local maximum or minimum.
- If it is known that $f(0) = 0$, sketch a possible graph of f.

Solution. Let's do this part by part.

- Since $f'(x) > 0$ on $(0, 1)$ and $(3, 5)$, our dictionary tells us $f(x)$ is increasing on this interval. Similarly, since $f'(x) < 0$ on $(1, 3)$ and $(5, 6)$, the dictionary tells us $f(x)$ is decreasing on this interval.
- By the first derivative test, $f(x)$ has a local maximum at $x = 1$ and $x = 5$ and a local minimum at $x = 3$.
- A sketch has been included above. When sketching it's useful to notice that the graph of $f'(x)$ tells us f is concave up on $(2, 4)$ and concave down on the intervals $(0, 2)$ and $(4, 6)$ (this follows because $f'(x)$ is increasing/decreasing on these intervals).

□

(2.9.3) Use the given graph of f to estimate the itervals on which the derivative $f'(x)$ is increasing or decreasing.

Solution. The calculus/geometry dictionary tells us that $f(x)$ is increasing whenever $f'(x) > 0$, and from the graph we see that this happens on approximately $(-1.5, 8.5)$. We spot where $f(x)$ is decreasing by finding when $f'(x) < 0$, which from the graph happens on the intervals $(-2, -1.5)$ and $(8.5, 9)$. □

(2.9.4)
- Sketch a curve whose slope is always positive and increasing.
- Sketch a curve whose slope is always positive and decreasing.
- Give equations for curves with these properties.

Solution. For these types of problems it's helpful to have the following figure in mind. From the sketch

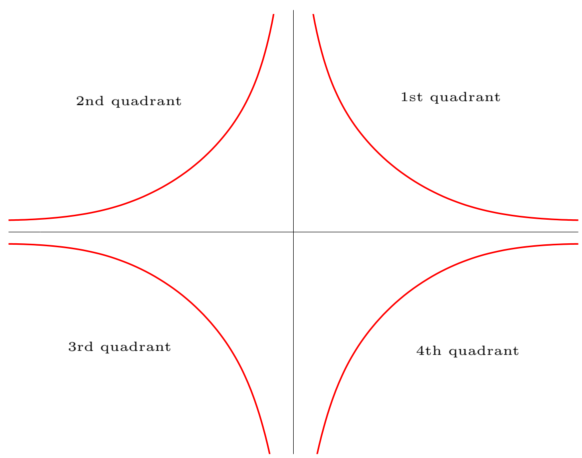


FIGURE 1. A handy sketch to remember

you can see that curve above the line $y = 0$ are concave up, and curves below $y = 0$ are concave down. You can also see curves in the 1st and 3rd quadrant have negative derivatives, while curves in the 2nd and 4th quadrant have positive derivatives.

- From what we've said, together with the fact that slopes of tangents to $f(x)$ (i.e., $f'(x)$) are increasing is equivalent to saying $f''(x) > 0$, we see that the curve in the 2nd quadrant fits the bill.
- Similarly, we see that a curve with positive, decreasing slopes (i.e., $f'(x) > 0$ while $f''(x) < 0$) is found in quadrant 4.
- The curves we've drawn in the previous solutions look like functions we've seen before. An example of a function that fits the bill for the first criterion is $f(x) = e^x$. An example of a function that fits the bill for the second is $f(x) = \ln(x)$.

□

(2.9.11) The graph of the derivative of $f'(x)$ of a continuous function $f(x)$ is shown.
- On what intervals is f increasing or decreasing?
- At what values of x does f have a local maximum or minimum?
- On what intervals is $f(x)$ concave upward or downward?
- State the x-coordinates of the points of inflection.
- Assuming that $f(0) = 0$, sketch the graph of f.

Solution. Let's do this step by step.
- Our calculus/geometry dictionary tells us that $f(x)$ is increasing on intervals where $f'(x) > 0$, and from the graph these intervals are $(0, 2)$, $(4, 6)$, and $(8, 9)$. $f(x)$ is decreasing on the intervals where $f'(x) < 0$, and from the graph these intervals are $(2, 4)$ and $(6, 8)$.

- Local maxima/minima can occur at critical points, which from the graph occur at $x = 2, 4, 6$, and 8. The first derivative test tells us that $x = 2$ and $x = 6$ are local maxima and that $x = 4$ and $x = 8$ are local minima.
- The calculus/geometry dictionary tells us that $f(x)$ is concave up whenever $f''(x) > 0$, which occurs (again using the dictionary) when $f'(x)$ is increasing. From the graph of $f'(x)$, we see this happens on the intervals $(3, 6)$ and $(6, 8)$. The function is concave down when $f'(x)$ is decreasing, which occurs on the interval $(0, 3)$.
- Concavity changes only at $x = 3$ (we just showed this!), so this is the only inflection point.
- Ok, we have *lots* of information about the curve. Let's use that information to sketch the curve.

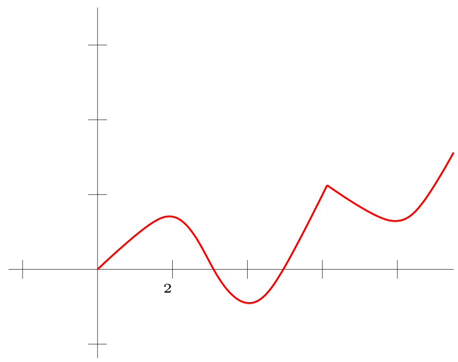


FIGURE 2. A very rough sketch of $f(x)$

□

(2.9.15) Sketch the graph of a function that satisfies all of the given conditions.

- $f'(0) = f'(4) = 0$
- $f'(x) > 0$ if $x < 0$
- $f'(x) < 0$ if $0 < x < 4$ or if $x > 4$
- $f''(x) > 0$ if $2 < x < 4$
- $f''(x) < 0$ if $x < 2$ or $x > 4$

Solution. The given criterion tell us that the function is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$. It also tells us f is concave up on the interval $(2, 4)$ and concave down on the intervals $(-\infty, 2)$ and $(4, \infty)$. It also needs to have slope zero at 2 and 4. With this information, we proceed to draw the graph of $f(x)$.

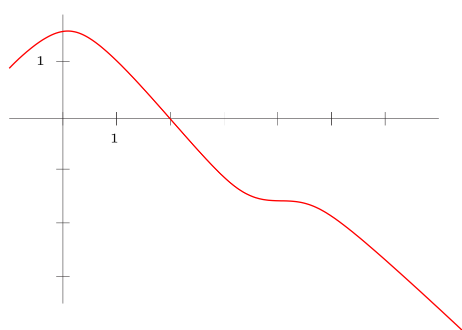


FIGURE 3. A possible sketch of $f(x)$

□

(2.9.21) Suppose that $f'(x) = xe^{-x^2}$.

- On what interval is f increasing? On what interval is f decreasing?
- Does f have a maximum or a minimum value?

Solution. Here we go.

- We know that f is increasing whenever $f'(x) > 0$. Now since $f'(x) = xe^{-x^2}$, and since $e^{\text{anything}} > 0$, we see that $f'(x) > 0$ whenever $x < 0$. Hence f is increasing on the interval $(-\infty, 0)$.
  In a similar way, $f'(x) < 0$ on the interval $(0, \infty)$, and so f is decreasing on this interval.
- From the first derivative test, we see that f has a maximum at $x = 0$. Since $x = 0$ is the only critical point, $x = 0$ is the only potential maximum/minimum value for the function, and so we're done.

$\square$

(2.9.24) Let $f(x) = x^4 - 2x^2$.

- Compute $f'(x)$ and $f''(x)$.
- On what intervals is f increasing or decreasing?
- On what intervals is f concave upward or downward?

Solution. Here we go.

- The power rule says $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$, and that $f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$.
- f is increasing whenever $f'(x) > 0$, which we se happens on the intervals $(-1, 0)$ and $(1, \infty)$. f is decreasing whenever $f'(x) < 0$, which happens on the intervals $(-\infty, -1)$ and $(0, 1)$.
- f is concave up whenever $f''(x) > 0$, which see see occurs on the intervals $(-\infty, \frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$. f is concave down whenever $f''(x) < 0$, which occus on the interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

$\square$

(2.9.25) The graph of a function f is shown. Which graph is an antiderivative of f and why?

Solution. The zeros of f correspond to places where the graph of the antiderivative of f is flat, so this implies the antiderivative is not a. Since f goes from negative to positive at its zero, the antiderivative should have a minimum by the first derivative test. This means b should be the antiderivative of f. $\square$

(2.9.26) The graph of a function f is shown. Which graph is an antiderivative of f and why?

Solution. The zeros of f correspond to places where the graph of the antiderivative of f is flat, which means c is not the antiderivative of f. Since f goes from positive to negative at its zero, the antiderivative should have a maximum by the first derivative test. This means a should be the antiderivative of f. $\square$

(E2) *Solution.* We take the equation $a^{\log_a(x)} = x$ and take the derivative of each side. On the right side this just gives 1, and on the left we use the chain rule to see that

$$\frac{d}{dx}\left[a^{\log_a(x)}\right] = \ln(a)a^{\log_a(x)}\frac{d}{dx}\left[\log_a(x)\right] = \ln(a)x\frac{d}{dx}\left[\log_a(x)\right].$$

This gives the equality

$$\ln(a)x\frac{d}{dx}\left[\log_a(x)\right] = 1,$$

which we can use to solve for $\frac{d}{dx}\left[\log_a(x)\right]$:

$$\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{\ln(a)x}.$$

$\square$

(E3.1) Completely analyze the function $f(x) = \sqrt[3]{x}$.

Solution. We know that we'll need to know information about the sign of the derivatives of f to analyze, so let's just record here that

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$$

and that

$$f''(x) = -\frac{2}{9}x^{-5}3 = -\frac{2}{9x^{\frac{5}{3}}} = -\frac{2}{9\sqrt[3]{x^5}}.$$

Now f is increasing when $f'(x) \geq 0$, and we can see that this occurs for *all* values of $x \neq 0$ (0 is the only critical point, and you can sample the derivative on either side by plugging in ± 1 into $f'(x)$ and determining the sign). This means f is increasing everywhere.

Further, since $x = 0$ is the only critical point (it's the only place where $f'(x)$ is undefined) and since $f'(x) > 0$ on either side of 0, this means that f has no local extrema.

Now we see that $f''(x) > 0$ on the interval $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$¡ so that $f(x)$ is concave up on $(0, \infty)$ and concave down $(-\infty, 0)$. This means that 0 is an (in fact, the only) inflection point. □

(E3.2) Completely analyze the function $f(x) = e^{x^2}$.

Solution. Again,we'll need the derivatives to analyze the function. The chain rule gives $f'(x) = 2xe^{x^2}$, and the product rule tells us $f''(x) = 4x^2e^{x^2} + 2e^{x^2} = e^{x^2}(4x^2 + 2)$.

Now since $e^{x^2} > 0$ for all x, the only critical point is $x = 0$. To the left of 0 we see that $f'(x) < 0$ (try plugging in -1, for example), and to the right of 0 we have $f'(x) > 0$. This means $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. The first derivative test therefore tells us that $x = 0$ is a local minimum.

Now since $f''(x) = e^{x^2}(4x^2 + 2)$, and since both e^{x^2} and $4x^2 + 2$ are always positive, this means $f''(x) > 0$ everywhere. This means $f(x)$ is concave up everywhere, and in particular there are no inflection points. □

(E3.3) Completely analyze the function $f(x) = x^3 - x$.

Solution. The derivatives comes quickly from the power rule: $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$.

The critical points are $x = \pm\frac{1}{\sqrt{3}}$. This helps us to see that $f'(x) > 0$ on the intervals $(-\infty, \frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$, so that f is increasing on these intervals. We also have $f'(x) < 0$ on the intervals $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, so that f is decreasing on this interval.

Using this information, the first derivative test tells us that $x = -\frac{1}{\sqrt{3}}$ is a local maximum and $x = \frac{1}{\sqrt{3}}$ is a local minimum.

Now the second derivative is zero only at $x = 0$, and clearly $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$. This means that f is concave up on $(0, \infty)$ and that f is concave down on $(-\infty, 0)$. This makes 0 the only inflection point. □

(E3.4) Completely analyze $f(x) = x\ln(x) - x$.

Solution. We know that $f'(x) = \ln(x)$ (see solutions to the last quiz) and that $f''(x) = \frac{1}{x}$.

Since $\ln(x) = 0$ only at $x = 1$, this makes $x = 1$ the only critical point. Using the graph of $\ln(x)$ we see that $f'(x) < 0$ on $(0, 1)$, so that f is decreasing on this interval. Similarly $f'(x) > 0$ on the interval $(1, \infty)$, so that f is increasing there. The first derivative test tells us that f has a minimum at $x = 1$.

The second derivative is positive everywhere on the domain of the function, and hence $f(x)$ is concave up everywhere. This means, in particular, that there are no inflection points. □