HOMEWORK 8 SOLUTIONS

(4.2.40) Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 4]$.

Solution. Since $f(x)$ is continuous on the given interval (after all, it's a polynomial), the strategy is to find the critical points of the function and then compare function values at critical points and endpoints.

Now $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$. Since this is defined everywhere, critical points occur when $f'(x) = 0$. Since $f'(x) = 3(x-1)(x-3)$, the critical points are 1 and 3. Now we see that

x	$f(x)$
-1	$(-1)^3 - 6(-1)^2 + 9(-1) + 2 = -1 - 6 - 9 + 2 = -14$
1	$1^3 - 6(1)^2 + 9(1) + 2 = 1 - 6 + 9 + 2 = 6$
3	$3^3 - 6(3)^2 + 9(3) + 2 = 27 - 54 + 27 + 2 = 2$
4	$4^3 - 6(4)^2 + 9(4) + 2 = 64 - 96 + 36 + 2 = 6$

Hence our function has a maxima at $x = 1$ and $x = 4$, and a minimum at $x = -1$. □

(4.2.42) Find the absolute maximum and absolute minimum values of $f(x) = (x^2 - 1)^3$ on $[-1, 2]$.

Solution. Since $f(x)$ is continuous on the given interval (it's a polynomial), the strategy is to find the critical points of the function and then compare function values at critical points and endpoints.

Now $f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2$. Since this is defined everywhere, critical points occur when $f'(x) = 0$, which happens with $6x = 0$ (so that $x = 0$) or $(x^2 - 1)^2 = 0$. Taking square roots and factoring gives $x = \pm 1$. Now we see that

x	$f(x)$
-1	$((-1)^2 - 1)^3 = 0$
0	$((0)^2 - 1)^3 = -1$
1	$((1)^2 - 1)^3 = 0$
2	$((2)^2 - 1)^3 = 27$

Hence our function has a maximum at $x = 27$ and a minimum at $x = 0$.

□

(4.2.43) Find the absolute maximum and absolute minimum values of $f(x) = t\sqrt{4 - t^2}$ on $[-1, 2]$.

Solution. Since $f(x)$ is continuous on the given interval (it's an algebraic function and the given interval is in its domain of definition), the strategy is the normal one: find the critical points of the function and compare function values at critical points and endpoints.

We see that

$$f'(x) = \sqrt{4-t^2} + t\left(\frac{1}{2\sqrt{4-t^2}}\right)(-2t) = \frac{4-t^2-t^2}{2\sqrt{4-t^2}}$$

. Critical points occurs where this function is 0 or undefined.

First, this function is 0 when the numerator is 0, which occurs when $4-2t^2=0$. One sees this occurs when $t=\pm\sqrt{2}$. Since $-\sqrt{2}$ is outside the interval, we ignore it.

This function is undefined when the denominator is 0, which occurs when $2\sqrt{4-t^2}=0$. Dividing by 2 on both sides and then squaring both sides gives the equation $4-t^2=0$, and this gives $t=\pm\sqrt{4}=\pm 2$. Since -2 isn't in the interval we ignore it. This means the critical points are $-\sqrt{2}$ and 2.

Now we compare the function values at endpoints and critical points.

t	$f(t)$
-1	$(-1)\sqrt{4-(-1)^2} = -\sqrt{3}$
$\sqrt{2}$	$\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2$
2	$2\sqrt{4-2^2} = 0$

Hence our function has a maximum at $x=\sqrt{2}$ and a minimum at $x=2$. □

(4.2.44) Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x}{x^2+4}$ on $[0,3]$.

Solution. As usual, $f(x)$ is continuous on the given interval (it's a rational function and the denominator never vanishes). We'll find the critical points of the function and compare function values at critical points and endpoints.

We see that

$$f'(x) = \frac{x^2+4-2x(x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

. Critical points occurs where this function is 0 or undefined. Since the denominator is never 0, we only need to know when the numerator is 0. But this occurs when $x=\pm 2$. The negative value is outside our interval, so we ignore it.

Now we compare the function values at endpoints and critical points.

x	$f(x)$
0	0
2	$\frac{2}{8}$
3	$\frac{3}{9+4} = \frac{3}{13}$

Since $\frac{2}{8} > \frac{3}{13}$, our function has a maximum at $x=2$ and a minimum at $x=0$. □

(4.2.48) Find the absolute maximum and absolute minimum values of $f(x) = x - \ln(x)$ on $[\frac{1}{2}, 2]$.

Solution. No surprise: $f(x)$ is continuous on the given interval. We'll find the critical points of the function and compare function values at critical points and endpoints.

We see that

$$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

. Critical points occurs where this function is 0 or undefined. Since the denominator is 0 only at 0 (which is outside our interval), we don't have any critical points arising in that fashion. Hence we only need to know when the numerator is 0. But this occurs when $x = 1$. Now we compare the function values at endpoints and critical points.

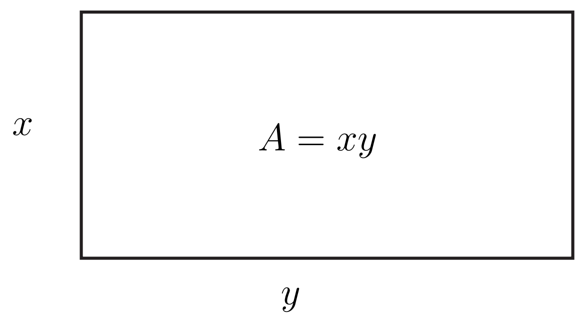
x	$f(x)$
$\frac{1}{2}$	$\frac{1}{2} - \ln\left(\frac{1}{2}\right) \approx 1.19$
1	$1 - 0 = 1$
2	$2 - \ln(2) \approx 1.31$

So our function has a maximum at $x = 2$ and a minimum at $x = 1$.

Note: You couldn't do those comparisons without a calculator, so I shouldn't have put this problem on the homework set. □

(4.6.5) Find the dimensions of a rectangle with perimeter 100m whose area is as large as possible.

Solution. Let's call x the height of the rectangle and y its width.



One can see that the resultant rectangle has area $A = xy$. We would like to maximize this function, but we don't know how to in its current form: it's a function of two variables, and we don't know how to maximize such a function. So what do we do?

The fact that we have 100 meters of perimeter comes to the rescue, since it tells us the perimeter of our figure is 100. Specifically, we have $100 = 2x + 2y$. We can now solve for y in terms of x and rewrite our area function: $100 = 2x + 2y \Leftrightarrow y = 50 - x$, and hence $A = xy = x(50 - x) = 50x - x^2$. This is the kind of function we can maximize, but first we need to know the domain of the function.

For this, notice that we could make a rectangle with no height (so that $x = 0$). This would be dumb, but we could do it. We could make another dumb rectangle: one without width, so that $x = 50$. The possible rectangles we could construct sit somewhere between these two stupid extremes, so we see that x lives in the interval $[0, 50]$. Now we're really in business since we have reduced our original problem to the following: maximize the function $A(x) = 50x - x^2$ on the interval $[0, 50]$.

Since $A(x)$ is continuous on $[0, 50]$ (it's a polynomial, so in fact it's continuous everywhere), we can find the extreme values of A by computing the value of of $A(x)$ at critical points and endpoints. We proceed to find critical points: $A'(x) = 50 - 2x$, and so the only critical point we have is $x = 25$.

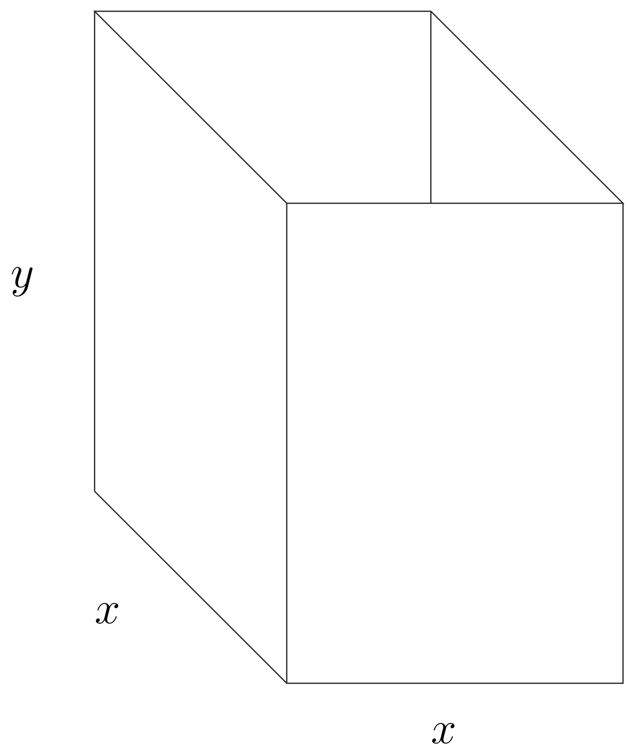
Now we need to evaluate $A(x)$ at the critical point and the endpoints of the interval.

x	$A(x)$
0	$A(0) = 0$
25	$A(25) = 625$
50	$A(50) = 0$

So to maximize the area of our rectangle we will make $x = 25$ (and so $y = 25$ since $y = 50 - x$), and the resultant area will be 625. □

(4.6.9) If 1200cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution. I'll draw my box like so: The volume of my box is $V = x^2y$, and the constraining equation



is $1200 = 4xy + x^2$. This means that $y = \frac{1200-x^2}{4x}$, and so $V(x) = x^2(\frac{1200-x^2}{4x}) = \frac{1}{4}(1200x - x^3)$. The interval we're working on is $[0, \sqrt{1200}]$ (the first corresponding to the base of your box being stupid (ie, the base has no width or length), and the second corresponding to your box having no height (when all the material is used to make the base of the box)). So we're trying to maximize the continuous function $V(x)$ on a closed interval $[0, \sqrt{1200}]$.

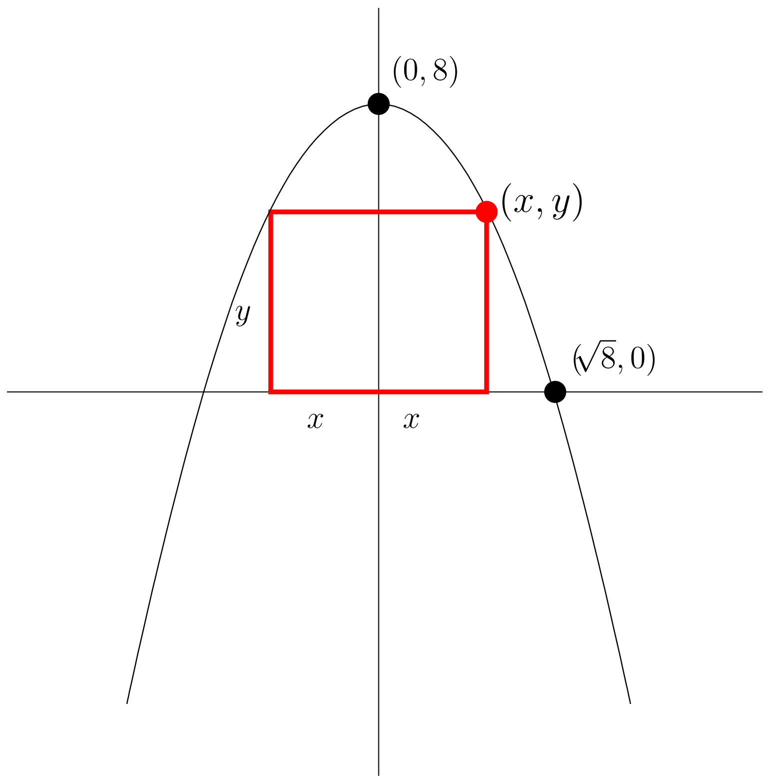
To solve this problem, we find critical points. Now $f'(x) = \frac{1}{4}(1200 - 3x^2)$. This means critical points occur at $x = \pm\sqrt{400} = \pm 20$. Since -20 is outside our interval, we ignore it. Now we compare function values at the critical point and at the endpoints.

x	$V(x)$
0	0
20	$\frac{1}{4}(1200(20) - 20^3) = \frac{1}{4}(24000 - 8000) = 4000$
$\sqrt{1200}$	0

Hence the largest possibe volume is 4000 (occuring when $x = 20$, and so $y = 20$). □

(4.6.16) Find the dimensions of the rectangle of largest area that has its base on the x-axies and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

Solution. My rectangle looks (approximately) like the following The area of my rectangle is $A = 2xy$,



and the constraining equation is $y = 8 - x^3$. This means that $A(x) = 2x(8 - x^2) = 16x - 2x^3$. The interval we're working on is $[0, \sqrt{8}]$ (the first corresponding to the x-coordinate of the point $(0, 8)$ and the second corresponding to the x-coordinate of the y-intercept, $(\sqrt{8}, 0)$). So we're trying to maximize the continuous function $A(x)$ on a closed interval $[0, \sqrt{8}]$.

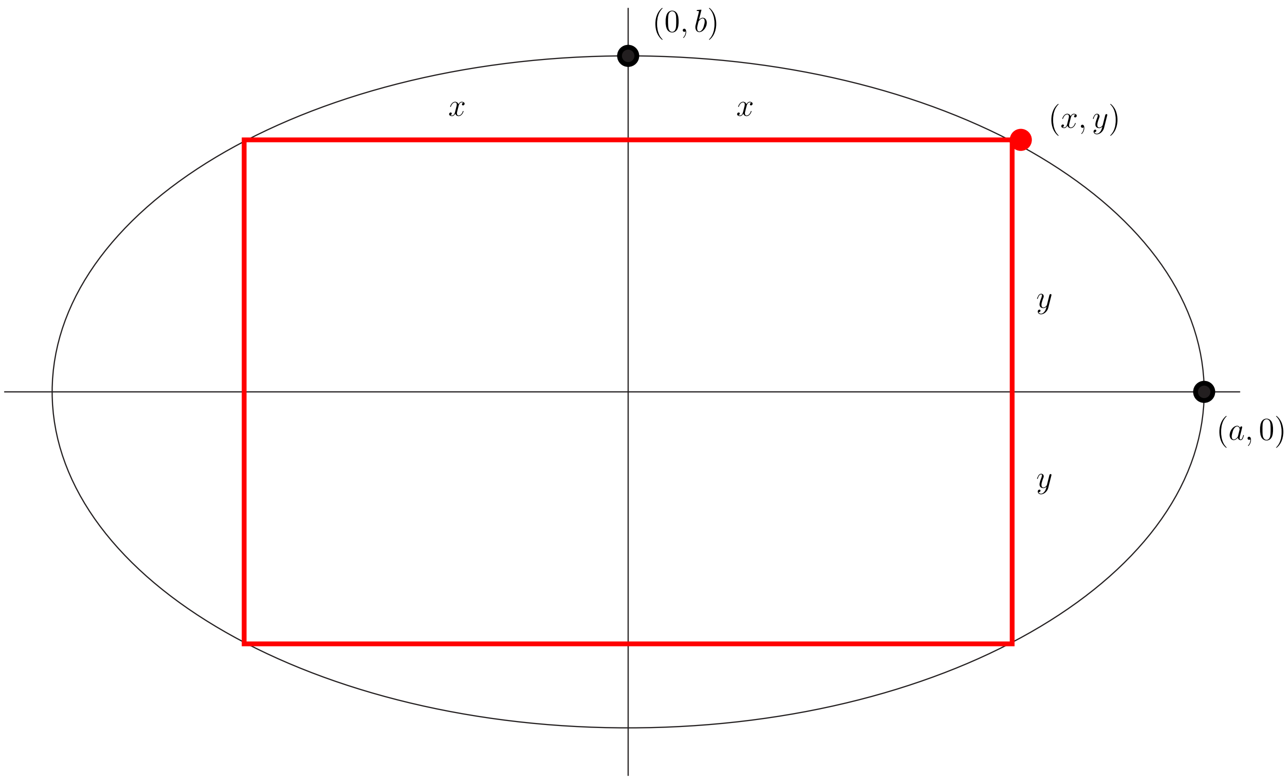
To solve this problem, we find critical points. Now $f'(x) = frm[o]--6 - 6x^2$. This means critical points occur at $x = \pm\sqrt{\frac{16}{6}} = \pm\frac{4}{\sqrt{6}}$. Since the negative value is outside our interval, we ignore it. Now we compare function values at the critical point and at the endpoints.

x	$A(x)$
0	0
$\frac{4}{\sqrt{6}}$	$16\frac{4}{\sqrt{6}} - 2(\frac{16}{6})^3 > 0$
$\sqrt{8}$	0

So our maximum occurs when $x = \frac{4}{\sqrt{3}}$ and $y = 8 - \frac{16}{6}$. This means our rectangle has width $\frac{8}{\sqrt{3}}$ and height $8 - \frac{16}{6}$. $\square$

(4.6.18) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. The picture looks like this The area of my rectangle is $A = 4xy$, and the constraining equation



is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This means $y = \sqrt{b^2(1 - \frac{x^2}{a^2})}$, so that

$$A(x) = 4x\sqrt{b^2(1 - \frac{x^2}{a^2})} = 4b\sqrt{x^2 - \frac{x^4}{a^2}}.$$

The interval we're working on is $[0, a]$ (the first corresponding to the x-coordinate of the point $(0, b)$ and the second corresponding to the x-coordinate of the y-intercept, $(a, 0)$). So we're trying to maximize the continuous function $A(x)$ on a closed interval $[0, a]$.

To solve this problem, we find critical points. Now

$$f'(x) = 4b\left(\frac{1}{2\sqrt{x^2 - \frac{x^4}{a^2}}}\right)\left(2x - \frac{4x^3}{a^2}\right) = \frac{4b(2x - \frac{4x^3}{a^2})}{2\sqrt{x^2 - \frac{x^4}{a^2}}}.$$

Critical points occurs where either the numerator or denominator is 0.

Now the numerator is 0 when $4b(2x - \frac{4x^3}{a^2}) = 0$. This means $x(2 - \frac{4x^2}{a^2}) = 0$, so that either $x = 0$ or $x = \pm\frac{a}{\sqrt{2}}$. The negative value is outside our interval, so we ignore it.

The denominator is 0 when $2\sqrt{x^2 - \frac{x^4}{a^2}} = 0$, which you can verify occurs only when $x = \pm a$. Again, $-a$ is outside the interval, so we ignore it. Now we compare function values at the critical point and at the endpoints.

x	$A(x)$
0	0
$\frac{a}{\sqrt{2}}$	$2ab$
a	0

The first and last quantities I know geometrically (they are the areas of the stupid rectangles), and the middle one I calculate using a little old-fashioned algebra. Hence the area of the largest rectangle I can inscribe in the ellipse is $2ab$. $\square$