HOMEWORK 9 SOLUTIONS

(3.8.9) Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

Solution. Since $f'(x) = -\frac{1}{2\sqrt{1-x}}$, this means $f'(0) = -\frac{1}{2}$. Together with the fact that $f(0) = 1$, this means the tangent line has equation

$$y = -\frac{1}{2}(x-0)+1.$$

Hence $\sqrt{0.9} \approx -\frac{1}{2}(0.1)+1 = 0.95$ and $\sqrt{0.99} \approx -\frac{1}{2}(0.01)+1 = 0.995$. □

(3.8.15) Use linear approximation to estimate $(2.001)^5$.

Solution. Let $f(x) = x^5$. Now we know that $f'(x) = 5x^4$, that $f'(2) = 80$ and that $f(2) = 64$. This means the tangent to the graph of $f(x)$ at 2 is given by the equation

$$y = 80(x-2)+64.$$

Hence we have

$$f(2.001) \approx 80(2.001-2)+64 = 64\frac{80}{1000} = 64\frac{2}{25}.$$

□

(3.8.16) Use linear approximation to estimate $e^{-0.015}$.

Solution. Let $f(x) = e^x$. Then we have $f'(0) = f(0) = 1$ (since $f'(x) = e^x = f(x)$), so the equation of the line tangent to $f(x)$ at 0 is

$$y = (x-0)+1 = x+1.$$

Hence we have

$$f(-0.015) \approx -0.015+1 = 0.985.$$

□

(4.6.3) Find two positive numbers whose product is 100 and whose sum is a minimum.

Solution. Let's call our postive numbers x and y. We're given that $xy = 100$, and we're told to minimize the quantity $S = x+y$. Using the equation $xy = 100$ gives $S = x+y = x+\frac{100}{x} = \frac{x^2+100}{x}$. Since x is any positive number, we're to optimize this function on the domain $(0,\infty)$.

Now we see that $S'(x) = 1-\frac{100}{x^2} = \frac{x^2-100}{x^2}$, and so the critical points are $x = 0$ and $x = \pm 10$. Only $x = 10$ is in our domain, so we can run the first derivative test for absolute extremes. In particular, since $S'(5) = 1-\frac{100}{25} = 1-4 < 0$ and $S'(100) = 1-\frac{100}{100^2} = 1-.01 > 0$, the first derivative tells us that $x = 10$ yields the absolute minimum value of $S(x)$. The corresponding value of y is $\frac{100}{10} = 10$. □

(4.6.4) Find a positive number such that the sum of the number and its reciprocal is as small as possible.

Solution. We're asked to minimize the function $f(x) = x+\frac{1}{x}$ on the interval $(0,\infty)$. Now we have $f'(x) = 1-\frac{1}{x^2} = \frac{x^2-1}{x^2}$, so the critical points of $f(x)$ are $x = 0$ and $x = \pm 1$. Again, $x = 1$ is the only critical point in the domain, so we can check to ensure that $x = 1$ minimizes $f(x)$ by running the first derivative test for absolute extrema.

Now we have $f'(\frac{1}{2}) = 1 - \frac{1}{1/4} = 1 - 4 < 0$ and $f'(2) = 1 - \frac{1}{4} > 0$, and so $x = 1$ does minimize the function $f(x)$. $\square$

(4.6.6) Find the dimensions of a rectangle with area 1000m^2 whose perimeter is as small as possible.

Solution. Let's call the dimensions of the rectangle x and y. We're told that $xy = 1000$, so that $y = \frac{1000}{x}$, and we're asked to minimize $P = 2x + 2y$. Now this gives $P(x) = 2x + 2\frac{1000}{x}$. We're to minimize this function on the domain $(0, \infty)$ (x has to be non-negative since it's a length, but can't be zero since we're supposed to have area 1000; it can take any positive value). So we compute $f'(x) = 2 - \frac{2000}{x^2} = \frac{2(x^2-1000)}{x^2}$, and note that the critical points of $f(x)$ are at $x = 0$ and $x = \pm\sqrt{1000}$. As usual, $\sqrt{1000}$ is the only critical point in the domain, so we run the first derivative test for absolute extrema.

Now $f'(10) = 2 - \frac{2000}{100} = 2 - 20 < 0$ and $f'(100) = 2 - \frac{2000}{10000} = 2 - \frac{1}{5} > 0$. The first derivative test tells us that $x = \sqrt{1000}$ minimizes the function $P(x)$ as desired. The corresponding y value is $\frac{1000}{\sqrt{1000}} = \sqrt{1000}$, and so the rectangle which minimizes perimeter is a square. $\square$

(4.6.10) A box with a square base and an open top must have a volume of $32,000\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

Solution. $\square$

(4.6.33) Find an equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant.

Solution. A line through $(3, 5)$ takes the form $y = m(x - 3) + 5$. The corresponding x and y intercepts are $(0, 5 - 3m)$ and $(\frac{-5+3m}{m}, 0)$, so we're asked to minimize the function

$$A(m) = \frac{1}{2}(5 - 3m)(\frac{-5 + 3m}{m}) = \frac{-9m^2 + 30m - 25}{2m}.$$

We're working on the interval $(-\infty, 0)$, since a line with 0 slope doesn't cut off any area from the first quadrant, and neither does a line with positive slope.

So we follow the usual procedure. We know that $A'(m) = \dfrac{2m(-18m + 30) - (2)(-9m^2 + 30m - 25)}{(2m)^2} = \frac{-18m^2+50}{4m^2}$. The critical points are therefore $x = 0$ (from when the denominator is zero) and $m = \pm\sqrt{50}18 = \pm\sqrt{25}9 = \pm\frac{5}{3}$. Since $\frac{5}{3}$ is the only critical point in our domain, we run the first derivative test. It will show us that $A'(m) < 0$ to the left of $\frac{5}{3}$ and $A'(m) > 0$ to the right, and hence $m = \frac{5}{3}$ minimizes area (as desired).

In the end, our line has equation

$$y - 5 = \frac{5}{3}(x - 3).$$

$\square$

(4.6.34) At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

Solution. We want to know when slope is maximial, and slope is given by $y' = 120x^2 - 15x^4$. We know that maxima occur at critical points, and since $y'' = 240x - 60x^3$, this tells us critical points are $x = 0$ and $x = \pm 2$. Now the first derivative test will tell us that the function has local maxima at $x = \pm 2$, but either of these could be the absolute maximum. But since $y'(2) = y'(-2)$, in fact both of these points are absolute maxima for the derivative, and hence y has maximum slope at the points ± 2. $\square$