MIDTERM 1 SOLUTIONS

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Name: ________________________________

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	15 pts	
2	20 pts	
3	25 pts	
4	30 pts	
5	35 pts	
6	30 pts	
Total	155 pts	

(1) (15pts) Complete each of the following sentences.

- Geometrically, the derivative $f'(a)$ is

the slope of the line tangent to the graph $y = f(x)$ at the point $(a, f(a))$.

- In class we defined a function $f(x)$ to be continuous at a point a if

$\lim_{x\to a} f(x) = f(a)$.

- Given a function $f(x)$ and a point $(a, f(a))$ on the graph of $f(x)$, the tangent problem asks

What is the equation of the line tangent to the graph $y = f(x)$ at the point $(a, f(a))$?

(2) (20 pts) Determine whether each statement is true or false for arbitrary functions $f(x)$ and $g(x)$. **If the statement is true, cite your reasoning. If it is false, provide a counterexample.**

- If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, then $\lim_{x\to 0} [f(x)g(x)]$ does not exist.

False. Let $f(x)$ and $g(x)$ be the functions defined by

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0 \end{cases} \qquad g(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0. \end{cases}$$

We have seen these functions in class before, and know that neither has a limit as $x \to 0$. However, we see that

$$f(x)g(x) = \begin{cases} 1 \cdot -1 = -1, & \text{if } x < 0 \\ -1 \cdot 1 = -1, & \text{if } x > 0, \end{cases}$$

and so $\lim_{x\to 0} [f(x)g(x)] = -1$. In particular, this limit exists.

- If $f(0)$ is undefined, then $\lim_{x\to 0} f(x)$ does not exist.

False. We said in class that, generally speaking, the value of the function at a point a has nothing to do with $\lim_{x\to a} f(x)$. In particular, the function can even be undefined at 0 and still have a limit. For instance, if we define $f(x) = 0$ for $x \neq 0$ and leave $f(0)$ undefined, then $\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0$.

(3) (25 pts)

- Draw the graph of a function $f(x)$ that satisfies $f'(0) = f'(10) = 0$, $f(0) = 5$, and $f(10) = 0$.

There are lots of possible answers. Here's one:

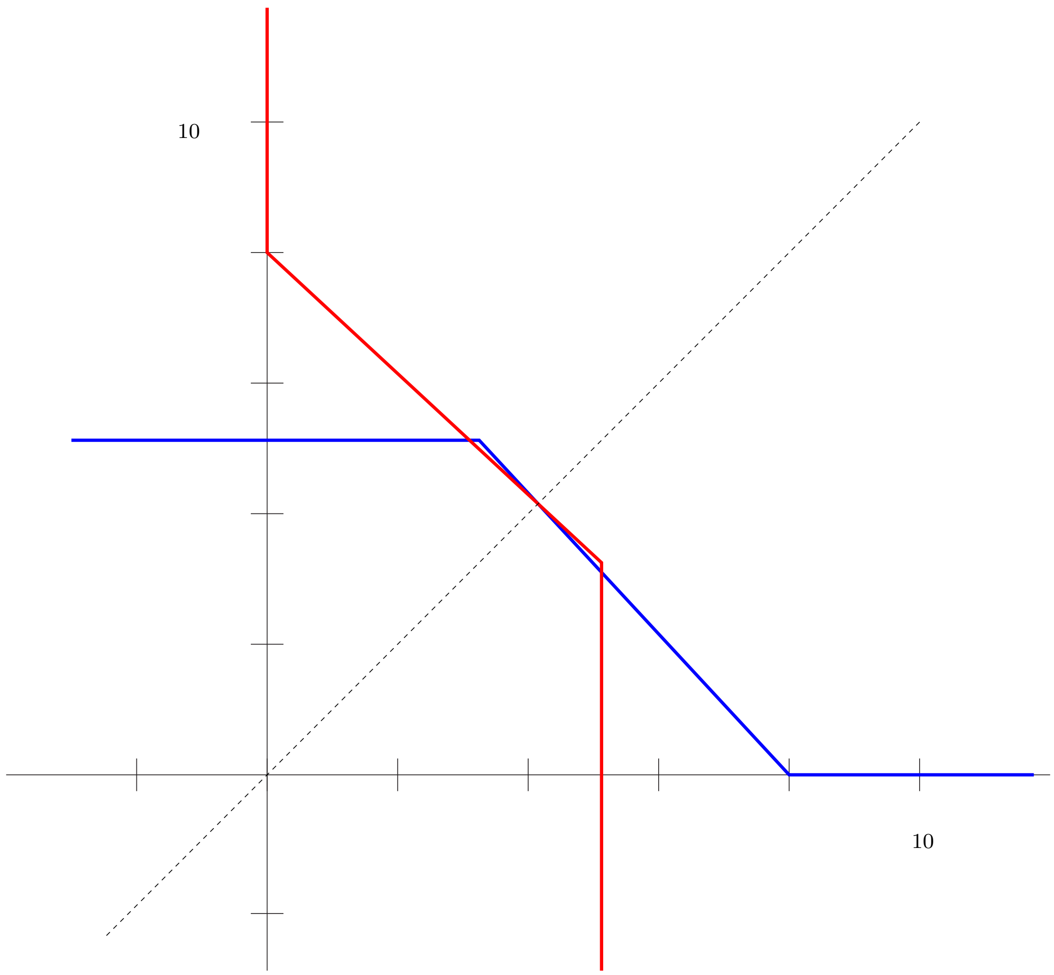

FIGURE 1. $f(x)$ in blue, $f^{-1}(x)$ in red

- On the same set of axes as above, graph $f^{-1}(x)$. Be sure to label which graph is $f(x)$ and which is $f^{-1}(x)$.

- Is $f^{-1}(x)$ a function? Explain why or why not.

For the function $f(x)$ drawn above, $f^{-1}(x)$ is not a function since it fails the vertical line test.

(4) (30 pts) Suppose $f(x)$ is a function that is continuous everywhere and satisfies $f(1) = -1$ and $f(5) = 10$.

- Is there a solution to the equation $f(x) = \log_5(x)$? If so, explain why. If not, give a counterexample.

A solution to the equation $f(x) = \log_5(x)$ is equivalent to a solution to the equation $g(x) = 0$, where $g(x) = f(x) - \log_5(x)$. Now we know that

$$g(1) = f(1) - \log_5(1) = -1 - 0 = -1 < 0$$
$$g(5) = f(5) - \log_5(5) = 10 - 1 = 9 > 0.$$

Since $g(x)$ is a continuous function (it's the difference of two continuous functions, $f(x)$ and $\log_5(x)$), and since $g(1) < 0$ and $g(5) > 0$, the intermediate value theorem says there is a solution to the equation $g(x) = 0$ on the interval $(1, 5)$. Hence there is a solution to the equation $f(x) - \log_5(x) = 0$ on the same interval, which just means there's a solution to the equation $f(x) = \log_5(x)$ on $(1, 5)$.

- Compute $\lim_{x\to 1} \left[f(x) + 2^{f(x)}\right]$.

Since $f(x)$ is continuous everywhere (in particular at 1), we know the function $f(x) + 2^{f(x)}$ is also continuous everywhere (it is the sum of two functions, the first continuous by hypothesis and the second the composition of two continuous functions, hence continuous). This means evaluating the limit at 1 is the same as evaluating the function at 1. We have

$$\lim_{x\to 1} \left[f(x) + 2^{f(x)}\right] = f(1) + 2^{f(1)} = -1 + 2^{-1} = 1 + \frac{1}{2} = -\frac{1}{2}.$$

(5) (35 pts) Let $f(x) = \dfrac{5x+2}{2x+1}$.

- Compute $f'(0)$ by evaluating an appropriate limit.

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0} \frac{\frac{5x+2}{2x+1} - \frac{2}{1}}{x} = \lim_{x\to 0} \frac{\frac{5x+2}{2x+1} - \frac{4x+2}{2x+1}}{x} \\ &= \lim_{x\to 0} \frac{5x+2-4x-2}{x(2x+1)} = \lim_{x\to 0} \frac{x}{x(2x+1)} = \lim_{x\to 0} \frac{1}{2x+1} = 1. \end{aligned}$$

- Write the equation of the line tangent to $y = f(x)$ at $(0, f(0))$.

Since $f'(0) = 1$ is the slope of the tangent line and $(0, 2)$ is a point on the tangent line, the equation is

$$y - 2 = 1(x-0) \quad \text{or, equivalently} \quad y = x + 2.$$

- Find $f^{-1}(x)$.

The procedure to find $f^{-1}(x)$ is to take the equation $y = f(x)$, swap x's and y's, and then solve for y. Swapping x and y gives $x = \dfrac{5y+2}{2y+1}$, and multiplying both sides of this equation by $2y+1$ yields the equation $x(2y+1) = 5y+2$. Expanding the left hand side gives $2yx + x = 5y + 2$, and now collecting all terms with a y onto the left and all terms without a y on the right gives

$$y(2x-5) = 2yx - 5y = 2 - x.$$

Dividing through by $2x - 5$ gives

$$f^{-1}(x) = \frac{2-x}{2x-5}.$$

(6) (30 pts)

- Evaluate $\lim_{x\to 2}\left[\dfrac{\left(\dfrac{x}{\sqrt{x+2}}-1\right)}{x-2}\right]$.

We solve this problem by first writing the difference of fractions as a single fraction, then rationalizing the numerator (via multiplication by conjugate, as per usual). Then it's a matter of factoring the numerator to get the cancellation we want.

$$\lim_{x\to 2}\left[\frac{\left(\frac{x}{\sqrt{x+2}}-1\right)}{x-2}\right] = \lim_{x\to 2}\left[\frac{x-\sqrt{x+2}}{(x-2)\sqrt{x+2}}\right] = \lim_{x\to 2}\left[\frac{x-\sqrt{x+2}}{(x-2)\sqrt{x+2}}\cdot\left(\frac{x+\sqrt{x+2}}{x+\sqrt{x+2}}\right)\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-x-2}{(x-2)(\sqrt{x+2})(x+\sqrt{x+2})}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x+1)}{(x-2)(\sqrt{x+2})(x+\sqrt{x+2})}\right]$$

$$= \lim_{x\to 2}\left[\frac{x+1}{(\sqrt{x+2})(x+\sqrt{x+2})}\right] = \frac{3}{8}.$$

- For $f(x) = (1+x)^2$, compute $f'(1)$ by evaluating an appropriate limit.

We'll evaluate this derivative the standard way: by plugging in the definition of the derivative and messing around with the numerator until we get some nice cancellation.

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+1+h)^2-(1+1)^2}{h} = \lim_{h\to 0}\frac{4+4h+h^2-4}{h}$$

$$= \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0}\frac{h(4+h)}{h} = \lim_{h\to 0}\,[4+h] = 4.$$