MIDTERM 2

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page or the scratch paper provided at the end of the exam. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Name: Solutions

Signature: Solutions

The following boxes are strictly for grading purposes. Please do not mark.

1	15 pts	
2	15 pts	
3	20 pts	
4	35 pts	
5	30 pts	
6	20 pts	
7	20 pts	
8	45 pts	
Total	200 pts	

(1) (15 pts) Complete each of the following sentences.

(a) The derivative of a function $f(x)$ is defined to be the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(b) The critical points of a function $f(x)$ are defined to be

those points where $f'(x)$ is either 0 or undefined.

(c) According to the calculus/geometry dictionary, the inequality $f''(x) > 0$ on (a, b) is equivalent to 2 things. They are

(i) the graph of $f(x)$ is concave up on (a, b) or
(ii) $f'(x)$ is increasing on (a, b)

(2) (15 pts) Determine whether each statement is true or false for arbitrary functions $f(x)$ and $g(x)$. **If the statement is true, cite your reasoning. If it is false, provide a counterexample.**

(a) If $f(x)$ satisfies $f'(0) = 0$, then $f(x)$ has a local maximum or local minimum at $x = 0$.

False. The function $f(x) = x^3$ has a critical point at 0 but neither a max nor min there (you can see this from the first derivative test or sketching the graph of $f(x)$).

(b) If a function is continuous at 0, then it is differentiable at 0.

False. Consider the absolute value function. It is continuous everywhere but not differentiable at 0 (since it has a cusp at 0).

(c) If $f'(a) = 0$, then $f''(a) = 0$ also.

False. Consider $f(x) = x^2$ and $a = 0$. Then $f'(a) = 0$, but $f''(a) = 2 \neq 0$.

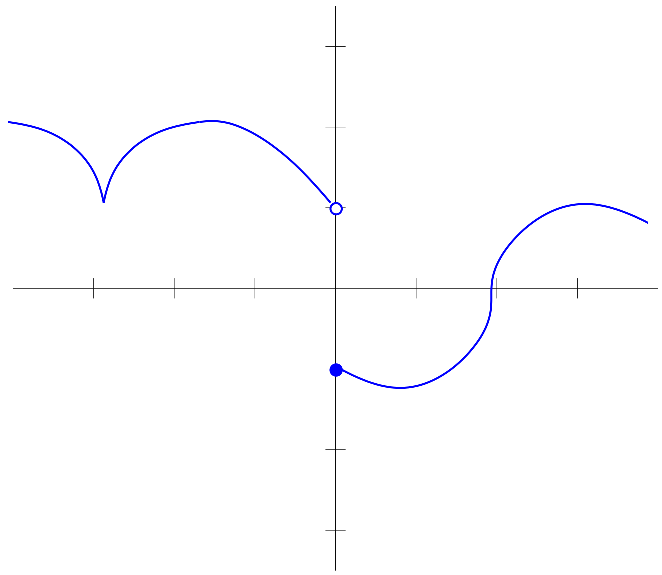
(3) (20 pts)

(a) Give an example of a function $f(x)$ which satisfies $f(x) = -f''(x)$.

The function $f(x) = \sin(x)$ satisfies this condition (as does $\cos(x)$).

(b) In class we said a function can fail to be differentiable at a point in three different ways. Draw the graph of a function which exhibits all of these failures, labeling each of the offending points by its failure.

The following function has a cusp at -3, a discontinuity at 0, and a vertical tangent at ≈ 2.



(4) (35 pts)

(a) Use the definition of the derivative as a limit to compute $\frac{d}{dx}[x^2]$.

By definition, we have

$$\frac{d}{dx}\left[x^2\right] = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

(b) Use the definition of the derivative as a limit to compute $\frac{d}{dx}[x]$.

By definition, we have

$$\frac{d}{dx}\left[x^2\right] = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1.$$

(c) Use your results from the previous two calculations, together with the chain rule, to prove that $\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$. You may not use the power rule. (Hint: Consider the equation $(\sqrt{x})^2 = x$.)

Applying $\frac{d}{dx}$ to the equation in the hint, we get 1 on the right side (from part (b) above) and $2\sqrt{x}\frac{d}{dx}\left[\sqrt{x}\right]$ on the left (this uses the chain rule and our computation $\frac{d}{dx}[x^2] = 2x$ from part (a)). Hence we have $2\sqrt{x}\frac{d}{dx}[\sqrt{x}] = 1$, and so

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}.$$

(5) (30 pts) Compute the following. You do not have to use the definition of the derivative as a limit.

(a) $\dfrac{d}{dx}\left[\dfrac{x^3 + x^2 + \ln(x)}{e^{\cos(x)}}\right]$

By the quotient and chain rules, we have

$$\frac{d}{dx}\left[\frac{x^3 + x^2 + \ln(x)}{e^{\cos(x)}}\right] = \frac{e^{\cos(x)}(3x^2 + 2x + \frac{1}{x}) - (x^3 + x^2 + \ln(x))(-\sin(x)\ e^{\cos(x)})}{e^{2\cos(x)}}$$

$$= \frac{3x^2 + 2x + \frac{1}{x} + \sin(x)(x^3 + x^2 + \ln(x))}{e^{\cos(x)}}.$$

(b) $\dfrac{d}{dx}\left[\arctan\left(\sqrt{1 + \arcsin(x)}\right)\right]$

The chain rule and our facts about derivatives of inverse trig functions gives

$$\frac{d}{dx}\left[\arctan\left(\sqrt{1 + \arcsin(x)}\right)\right] = \frac{1}{1 + (\sqrt{1 + \arcsin(x)})^2}\ \frac{1}{2\sqrt{1 + \arcsin(x)}}\ \frac{1}{\sqrt{1 - x^2}}$$

$$= \frac{1}{2 + \arcsin(x)}\ \frac{1}{2\sqrt{1 + \arcsin(x)}}\ \frac{1}{\sqrt{1 - x^2}}.$$

(6) (20 pts) For a particular function $f(x)$, the graph of $f'(x)$ is:

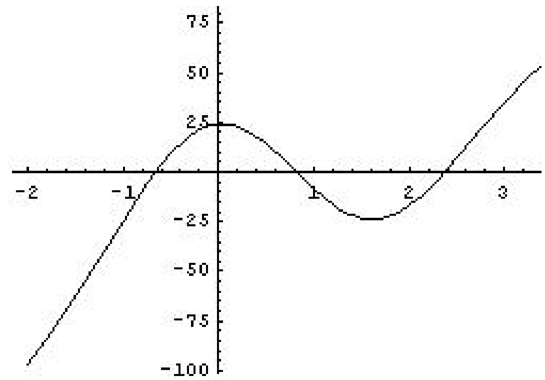
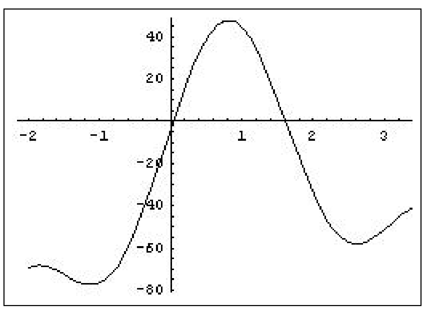
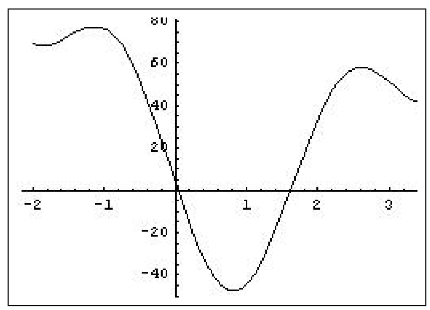
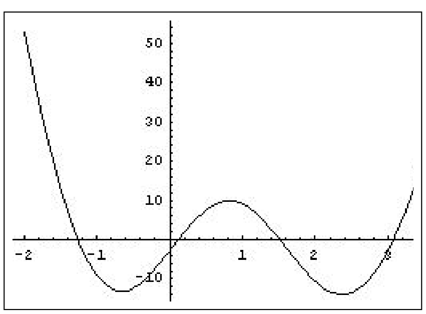


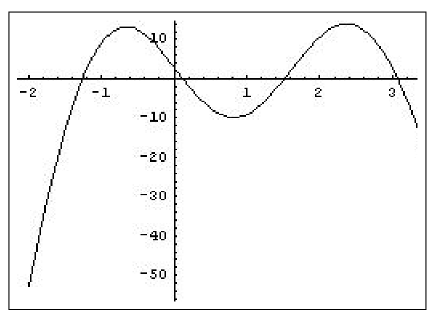
FIGURE 1. The graph of $f'(x)$

Which of the following 4 graphs could be the graph of $f(x)$? Why? An answer without sufficient justification will receive little or no credit.

(A)


(B)


(C)


(D)


Since the graph of $f'(x)$ is negative to the left of -0.6, the graph of $f(x)$ must be decreasing to the left of -0.6. This eliminates graphs A and D. Now $f'(x)$ is zero in 3 places, but the graph of B is flat in four places, and so $f(x)$ can't be given by the graph of B. Hence the answer must be C.

(7) (20 pts) Find the equation of the line tangent to $x^3 - y^3 = -6xy$ at $(3, -3)$.

We need to know the slope of the tangent at $(3, -3)$, and we know slopes are represented by the quantity y'. To find y' we use implicit differentiation. The left side has derivative $3x^2 - 3y^2y'$, and the right has derivatives $-6y - 6xy'$. Combining these two equations and solving for y' gives

$$y' = \frac{3x^2 + 6y}{3y^2 - 6x}.$$

Plugging in $(3, -3)$ gives us the slope of the tangent to the graph at this point:

$$m = \frac{3(3)^2 + 6(-3)}{(-3)^2 - 6(3)} = 1.$$

Therefore the equation of the line tangent to the graph is

$$y + 3 = 1(x - 3).$$

(8) (45 pts) Let $f(x) = 4x^5 + 5x^4 - 40x^3 + 17$.

(a) Compute $f'(x)$ and $f''(x)$. You do not have to use the definition of the derivative as a limit.

The power rule gives $f'(x) = 20x^4 + 20x^3 - 120x^2$ and $f''(x) = 80x^3 + 60x^2 - 240x$.

(b) List all possible extreme values of the function.

The possible extreme values are the critical points of $f(x)$. Since $f'(x) = 20x^2(x^2 + x - 6) = 20x^2(x + 3)(x - 2)$, we see that the critical points are $x = -3, 0, 2$.

(c) Find intervals on which $f(x)$ is increasing and intervals on which $f(x)$ is decreasing.

Since $f'(-4) = 20(-4)^2(-1)(-6) > 0$, we see that $f(x)$ is increasing on $(-\infty, -3)$. Since $f'(-1) = 20(-1)^2(2)(-3) < 0$ we have $f(x)$ is decreasing on $(-3, 0)$, and similarly since $f'(1) = 20(1)^2(4)(-1) < 0$ we have $f(x)$ is decreasing on $(0, 2)$. (You could say that $f(x)$ is decreasing on $(-3, 2)$). Finally, since $f'(3) = 20(3)^2(6)(1) > 0$ we see that $f(x)$ is increasing on $(2, \infty)$.

(d) From your list of possible extreme values, determine which are actually local maxima and which are actually locally minima (for each, be sure to justify why it is a max/min).

Our previous calculations tell us that $x = -3$ is a local maximum and $x = 2$ is a local minimum (note: our previous calculations amount to using the 1st derivative test). It also says that $x = 0$ is neither a local max nor a local minimum.

(e) I was originally planning on asking you to find intervals on which $f(x)$ is concave up and intervals on which $f(x)$ is concave down. If I had asked that question, how would you have solved it?

We begin by finding those places where $f''(x)$ is zero or undefined. Marking these spots on our 'second derivative line,' we sample $f''(x)$ on the intervals defined between these points to determine the sign of $f''(x)$ on said intervals. If $f''(x) > 0$ on an interval, $f(x)$ is concave up on this interval; similarly $f''(x) < 0$ on an interval if $f(x)$ is concave down there.