PRECALCULUS REVIEW

1. Functions

In this section we'll restate the basic classes of functions we'll be discussing in this class, see some examples, review important identities, and discuss inverses. Students who feel comfortable with standard mathematical functions might still benefit from reviewing the section on inverses.

1.1. Algebraic functions.

An algebraic function is any function that can be written by taking finite sums, products, and/or roots of a variable x. The simplest algebraic functions are the polynomials, which are sums of powers of the variable. The domain of all polynomials is $(-\infty, \infty)$. Some examples of polynomials include

$$f_1(x) = x \qquad f_2(x) = x^2$$
$$f_3(x) = 2x^7 - 32x^5 + 11 \qquad f_4(x) = 17$$

Of course, not all algebraic functions are polynomials. Also, not all algebraic functions are defined on the entire real line, and so one must be careful to consider the domain of a general algebraic function. Some algebraic functions and their maximum domains include

Function	Domain
$g_1(x) = \sqrt{x}$	$[0, \infty)$
$g_2(x) = \dfrac{1}{x-1}$	$\{x \in \mathbb{R} : x \neq 1\}$
$g_3(x) = \sqrt[3]{2x^7 - 32x^5 + 11}$	$(-\infty, \infty)$
$g_4(x) = (x+2)^{\frac{3}{2}}$	$[-2, \infty)$

1.2. Trigonometric functions.

Trig functions are those functions which involve the functions $\sin x$ or $\cos x$. There are other standard trigonometric functions (tangent, cotangent, secant, and cosecant), but they all arise by taking products and quotients of these orginal two.

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x}$$

Note: Sine and Cosine are functions of a variable, so the variable must appear somewhere whenever you write a trigonometric function.

Right	**Wrong**
$\sin x$	$\sin$
$\sin^2 x + \cos^2 x$	$\sin^2 + \cos^2$

Also, when the input for a trig function is 'big,' one usually will write this input inside parentheses. Thus '$\sin(x + 3\pi)$' means "the value of sine at $x + 3\pi$."

1.3. **Trigonometric identities.** There are several key equations which allow one to simplify expressions involving trigonometric functions. Essentially these identities come from either the periodicity of $\sin x$ and $\cos x$, or the equation

$$\sin^2 x + \cos^2 x = 1.$$

Here are a few that you'll want to keep in mind:

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

$$\csc^2 x - \cot^2 x = 1 \qquad \sec^2 x - \tan^2 x = 1$$

1.4. **Exponential functions.** When b is a positive integer and a is a real number, the symbol a^b denotes the product of a with itself b times:

$$a^b = \underbrace{a \cdots a}_{\text{b times}}.$$

What does a^b mean when b is not a positive integer? When b is a positive fraction, say $b = c/d$, we define

$$a^b = a^{c/d} = \sqrt[d]{a^c}.$$

When $b < 0$, we define

$$a^b = \frac{1}{a^{|b|}}.$$

In this way we can define a^b for any real number b.

Exponential functions are those functions which have a variable (or some function of a variable) as the exponent of some base. Here are some examples:

$$h_1(x) = 2^x \qquad h_2(x) = 11^{x^2+1} \qquad h_3(x) = 4^{2^x}$$

$$h_4(x) = e^x \qquad h_5(x) = e^{\sin x} \qquad h_6(x) = e^{-x}$$

In the last few examples, the number e represents the so-called natural base. It is a number in the neighborhood of 2.7, but its exact value is not so important. We will see later in the term that e has a lot of very nice properties, so it will become one of our best friends.

1.5. **Exponent identities.** There are a few useful rules about exponents which should be in your comfort zone.

- $a^b \cdot a^c = a^{b+c}$
- $\left(a^b\right)^c = a^{bc}$
- $a^{-b} = \dfrac{1}{a^b}$

1.6. **Inverse functions.** There are several ways of putting together 2 old functions to make a new function. For instance, one might add two functions together or divide one function by another. A little trickier way of putting two functions together is *function composition*:

$$(f \circ g)(x) = f(g(x)).$$

To evaluate $f \circ g$ at x, one first evalues g at x, and then plugs this value into f. For instance, if $f(x) = x^2 + 1$ and $g(x) = 2^x$, then $f(g(2)) = g(2)^2 + 1 = 4^2 + 1 = 17$ and $g(f(2)) = 2^{f(2)} = 2^{2^2+1} = 2^5 = 32$. As you can see, in general $f \circ g \neq g \circ f$.

Definition. For a function $f : A \to B$, the inverse of f, denoted f^{-1}, is defined as follows: $f^{-1}(x) = \{a \in A : f(a) = x\}$. In words, $f^{-1}(x)$ returns the set of all inputs a for which $f(a) = x$. Notice that this means the domain of f^{-1} is the range of f (i.e., the set of all outputs of the function f).

Warning: The symbol $f^{-1}(x)$ could also be interpretted to mean $\frac{1}{f(x)}$. However, in general $f^{-1}(x) \neq \frac{1}{f(x)}$. Don't get these two things confused!

Warning: It is not always true that the inverse of a function is a function! For instance, $f(x) = x^2$ has $f^{-1}(4) = \{-2, 2\}$.

Here are some standard functions and their inverses.

$f(x)$	Domain(f)	$f^{-1}(x)$	Domain(f^{-1})
x^2	$(-\infty, \infty)$	$\sqrt{x}$	$[0, \infty)$
x^3	$(-\infty, \infty)$	$\sqrt[3]{x}$	$(-\infty, \infty)$
$\sin x$	$(-\infty, \infty)$	$\arcsin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos x$	$(-\infty, \infty)$	$\arccos x$	$[0, \pi]$
e^x	$(-\infty, \infty)$	$\log x = \ln x$	$(0, \infty)$
a^x	$(-\infty, \infty)$	$\log_a x$	$(0, \infty)$

Coming out of these functions are a few important identities, but you'll want to hold the following two particularly close to your heart.

Identity	Domain
$e^{\log x} = x$	$(0, \infty)$
$\log(e^x) = x$	$(-\infty, \infty)$

1.7. **Evaluating functions.** You should feel comfortable evaluating functions. If you don't, see the professor for extra practice with this concept.

Often in evaluating functions you will wind up with a quantity which won't simplify. This is ok! For instance, if $f(x) = e^{\sqrt{x}}$ and you're asked to evaluate $f(3)$, you would write $f(3) = e^{\sqrt{3}}$. If you had a calculator, you might try approximating this value to a few decimals. Since we're not using calculators on exams or quizzes, however, this decimal approximation is unnecessary.

Later in the course you will be expected to know the values of $\sin x$ and $\cos x$ at certain standard values. A great way to remember these and other important values of trig functions is with the unit circle. On the unit circle, x values stand for cosine and y values stand for sine. If you have never used the unit circle before and have a hard time remembering important values of sine and cosine, see the instructor for extra help.

2. Graphs of functions

You will be expected to recognize and recreate graphs of all your favorite functions. These include (but aren't limited to) the graph of a generic line $y = mx + b$, the graph of parabolae (quadratic polynomials), cubics, basic trig functions, exponential functions, and logarithms.

Given the graph of a function $f(x)$, you will also be expected to be able to sketch the graph of a functions like $f(x+1), f(2x), f(x) - 2, 3f(x)$, and $3f(x+1) - 2$. Some nice examples of this can be found in your book, section 1.3.

Given the graph of a function $f(x)$, it is also useful to be able to draw a graph of its inverse, $f^{-1}(x)$. To do this, 'reflect' the given graph across the line $y = x$.

3. Solving Equations

In this class and its sequel, it will be essential that you can solve equations. That is, given an function $f(x)$, you should be able to find all values of x for which $f(x) = 0$.

3.1. **Factoring.** Since a product of numbers is zero if and only if at least one of the factors is zero, solving the equation $f(x) = 0$ is greatly simplified if you can factor $f(x)$ as a product of functions. For instance, if I'm asked to solve the equation $x^2 \sin x - x \sin x = 0$, I first rewrite the equation as

$$x \cdot (x - 1) \cdot \sin x = 0.$$

Hence, my function will be zero exactly at those values of x where either $\sin x = 0$, $x = 0$, or $x - 1 = 0$. In this case, this means the solutions to the given equation are $x = 0$, $x = 1$, and all integer multiples of π.

3.2. **Quadratic formula.** One type of equation which you will likely run in to frequently is a quadratic equation. A quadratic equation is of the form $ax^2 + bx + c = 0$. Often you will be able to factor the polynomial to find solutions quickly, but you should also be prepared to find solutions when you cannot factor the polynomial. When $b^2 - 4ac \geq 0$, the solutions are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac < 0$, the equation has no solutions.

4. Precal Review

4.1. **Slope.** Given two points in the plane, you should feel comfortable finding the slope of the line passing through these two points. If the points are (x_1, y_1) and (x_2, y_2), then the slope is defined as

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Some people remember this as 'rise over run.'

4.2. **Lines.** There are several ways to write the equation for a line, and all are useful to remember. At the very least, however, you should feel very comfortable with one of the methods for writing a line. For the two styles shown below, m is the slope of the line, (x_1, y_1) is a fixed point on the line, and b is the y-intercept of the line (that is, $(0, b)$ is a point on the line).

Point-slope form. $y - y_1 = m(x - x_1)$

Slope-intercept form. $y = mx + b$

Notice that the first comes directly from the definition of the slope of the line, where we've replaced y_2 by y and x_2 by x. Also notice that the slope-intercept form comes from the point-slope form by solving for y.