# Fundamental Theorem

# Last Time

<u>Thm</u> (Equivalent Characterizations of intermediate Galoisness)
Let $E/F$ be Galois, and let $F \leq K \leq E$. Then the following are equivalent.

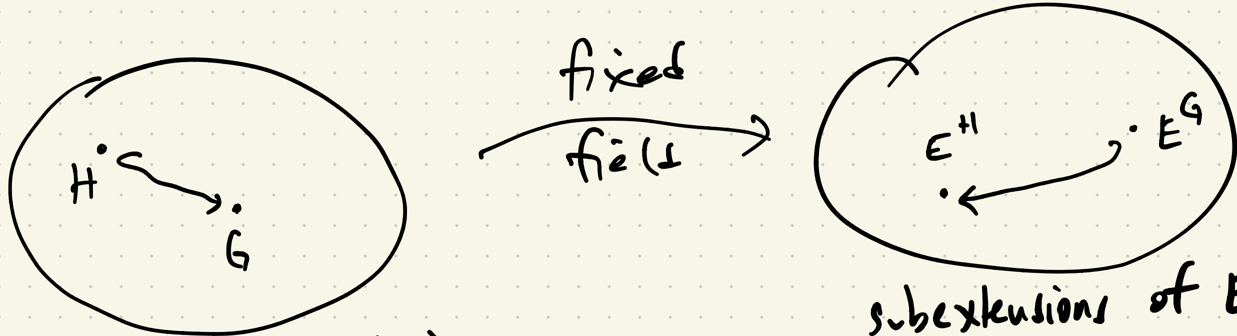
(i) $\mathrm{Conj}(K) = \{K\}$

(ii) For all $\sigma \in \mathrm{Gal}(E/F)$, we have $\sigma|_K \in \mathrm{Gal}(K/F)$

(iii) $\mathrm{Gal}(E/K) \trianglelefteq \mathrm{Gal}(E/F)$

(iv) $K/F$ is Galois.

# Even Further Back




~~subsets~~ subgroups of Aut(E)

subextensions of E

**Properties**

- contravariance
- preserves "bigness"
- injective

---

**New Stuff**

Are these universes "the same"?

## Notation / Basic Assumptions

Let $E/F$ be Galois, and let $G = \mathrm{Gal}(E/F)$

Let $\mathrm{Sub}(G)$ be the set of subgroups of $G$.

Let $\mathrm{Lat}(E/F)$ be the set of intermediate fields in $E/F$.

Define $\mathcal{F} : \mathrm{Sub}(G) \to \mathrm{Lat}(E/F)$ be $\mathcal{F}(H) = E^H$

and $\mathcal{G} : \mathrm{Lat}(E/F) \to \mathrm{Sub}(G)$ be $\mathcal{G}(K) = \mathrm{Gal}(E/K)$

$$\begin{tikzcd}
\mathrm{Sub}(G) \arrow[r, bend left, "\mathcal{F}"] & \mathrm{Lat}(E/F) \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Thm (Fundamental Theorem of Galois Theory)

The functions $\mathcal{F}$ and $\mathcal{G}$ satisfy $\mathcal{G} \circ \mathcal{F} = id_{Sub(G)}$

and $\mathcal{F} \circ \mathcal{G} = id_{Lat(E|F)}$, and are both contravariant.

Furthermore, we have

(i) for all $H \in Sub(G)$, then $[\mathcal{F}(H) : F] = \dfrac{|G|}{|H|}$

(ii) for all $K \in Lat(E|F)$, then $\dfrac{|G|}{|\mathcal{G}(K)|} = [K : F]$

(iii) $K/F$ is Galois iff $\mathcal{G}(K) \trianglelefteq G$

(iv) $H \trianglelefteq G$ iff $\mathcal{F}(H)/F$ is Galois.

Pf We've already seen $\mathcal{F}$ is contravariant, and by hwk 9, problem 4 we get $\mathcal{G}$ is contravariant.

Now, let's show $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathrm{Sub}(G)}$. Let $H \leq G$ be given, and we'll show $(\mathcal{G} \circ \mathcal{F})(H) = H$.

Note $(\mathcal{G} \circ \mathcal{F})(H) = \mathcal{G}(\mathcal{F}(H)) = \mathcal{G}(E^H) = \mathrm{Gal}(E/E^H)$.

It's certainly true that any $\sigma \in H$ has $\sigma \in \mathrm{Gal}(E/E^H)$ since $\sigma \in H$ implies $\sigma \in \mathrm{Aut}(E)$, and since $E^H = \{e \in E : h(e) = e \text{ for all } h \in H\}$ we get $\sigma$ fixes $E^H$.

So we get $H \leq \mathrm{Gal}(E/E^H)$.

On the other hand,

$$|\mathrm{Gal}(E/E^H)| = [E : E^H] = |H|$$

since $E/E^H$ is Galois

Since $|\mathrm{Gal}(E/E^H)|$ is finite, we get $H = \mathrm{Gal}(E/E^H)$
$= (\mathcal{G} \circ \mathcal{F})(H)$

Now, let's show $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathrm{Lat}(E/F)}$. Let $F \leq K \leq E$.

Then $(\mathcal{F} \circ \mathcal{G})(K) = \mathcal{F}(\mathcal{G}(K)) = \mathcal{F}(\mathrm{Gal}(E/K)) = E^{\mathrm{Gal}(E/K)}$

Since $E/K$ is Galois, we get $E^{\mathrm{Gal}(E/K)} = K$.

since $G = \mathrm{Gal}(E/F)$ and $E/F$ Galois

Now for

(i) $$\frac{|G|}{|H|} = \frac{[E:F]}{|H|} = \frac{\cancel{[E:E^H]}[E^H:F]}{\cancel{|H|}} = [E^H:F]$$

$$= [\mathfrak{F}(H):F].$$

(ii) $$\frac{|G|}{|\mathcal{G}(K)|} = \frac{|G|}{|\mathrm{Gal}(E/K)|} = \frac{[E:F]}{[E:K]} = [K:F].$$

(iii) & (iv) are the content of "Equivalent Characterizations of intermediate Galoisness".

$\blacksquare$