ARVESON'S WORK ON ENTANGLEMENT

FRED SHULTZ

Abstract. We summarize William Arveson's work on entanglement in quantum information theory.

William Arveson [1934-2011] worked mostly in the area of operator theory and operator algebras, but starting in 2008 turned to the area of quantum information theory, and wrote several papers relating to entanglement. The purpose of this article is to make better known to the mathematical community his work in this area.

We will define entanglement mathematically below, but first give a physical description. Entanglement is a property of physical systems (at a small, "quantum" scale) and their subsystems. When two physical systems interact, the physical state of the interacting systems contains more information than the individual subsystems. Furthermore, this feature (called entanglement) persists even if the subsystems become separated by a large distance. In recent years, it has been realized that entanglement has remarkable applications in quantum computing, quantum cryptography, and quantum communication.

Now we briefly summarize the main results of Arveson's papers [2, 3, 4]. In the first paper, Arveson shows that the state of a physical system is almost always entangled if it has low rank compared to its rank on a subsystem. In the second paper, he shows that physical operations ("quantum channels") satisfy a dichotomy: they either always destroy entanglement ("entanglement breaking") or almost always preserve entanglement. Furthermore, if the quantum channel has low rank, it almost always preserves entanglement. The third paper concerns states which are maximally entangled. There is a well-accepted definition of that term for the simplest physical systems (pure states on bipartite systems), but not more generally. Arveson proposes a general definition, and gives an explicit description of maximally entangled states in many cases.

Date: November 10, 2013.

2000 *Mathematics Subject Classification.* Primary 46N50, 46L30; Secondary 81P68, 94B27.

Key words and phrases. entanglement, quantum information.

Now we turn to a more detailed description of the papers. We begin with mathematical terminology relevant to the first two papers. The key physical notions are observables (quantities that can be measured), states (giving expectation values for observables), and quantum channels (physical operations on states), and we now define the mathematical equivalents.

An observable is an Hermitian matrix in the space M_n of $n \times n$ complex matrices. A *state* is a *density matrix* (a positive semidefinite matrix d of trace 1), identified with the positive linear functional that takes $x \in M_n$ to $\mathrm{tr}(xd)$. For each unit vector $\xi \in \mathbb{C}^n$, the *vector state* ω_ξ is the functional $\omega_\xi(x) = \langle x\xi, \xi \rangle$, and the vector ξ, the associated density matrix, and the vector state ω_ξ all are referred to as *pure states.* (They are the extreme points of the convex set of states.)

Observables (and states) for two interacting physical systems are represented by matrices in $M_m \otimes M_n$, which can be identified as $m \times m$ block matrices with entries in M_n, hence with matrices in M_{mn}. The matrices for observables for one of the individual systems live in $M_m \otimes I$ (identified with M_m), and for the other system live in $I \otimes M_n \cong M_n$.

A physical operation should be a linear map Φ that takes states to states, hence is a positive map and preserves trace. Furthermore, if we have two interacting systems (say belonging to Alice and Bob), and an operation $\Phi : M_n \to M_p$ is carried out on Bob's system, and nothing is done to Alice's system M_k, the operation on the combined system is represented by $I \otimes \Phi$, and this should also take states to states and hence be positive. Thus Φ should have the property that $I_k \otimes \Phi : M_k \otimes M_n \to M_k \otimes M_p$ is a positive map for all $k \geq 1$. This is precisely the definition of a *completely positive map.* Hence as pointed out by Kraus [7] a physical operation should be a completely positive trace preserving map; such maps are called *quantum channels.*

(The notion of completely positive maps was due originally to Stinespring [12] in 1955, and didn't attract much attention for quite a few years. Then Arveson realized that Stinespring's notion was exactly what he needed in his study of multivariable operator theory and nonself adjoint operator algebras. Arveson's famous extension theorem for completely positive maps was proven in [1] in 1969, and plays a central role in that paper, and in operator theory and operator algebras thereafter.)

A product state $\sigma \otimes \tau$ represents a bipartite state where there is no interaction between the two systems. A convex sum of such product states represents a mixture of non-interacting systems, and is said to be *separable.* A state is *entangled* if it is not separable. (A pure state

ω_ξ is separable iff ξ is a product vector $\eta \otimes \nu$ for some $\eta \in \mathbb{C}^m$ and $\nu \in \mathbb{C}^n$, and is entangled for all other vectors $\xi \in \mathbb{C}^m \otimes \mathbb{C}^n$.)

We now describe the paper "The probability of entanglement". Arveson's main result (assuming $m \geq n$) states that if a state ρ on $M_m \otimes M_n$ extends a state ω on M_n, and $\operatorname{rank} \rho \leq (1/2) \operatorname{rank} \omega$, then ρ is almost surely entangled. (Using different techniques and a different probability measure, this has been improved by Ruskai and Werner [10] to $\operatorname{rank} \rho \leq \operatorname{rank} \omega$.) Arveson also shows that the probability that an extension of ω of maximal rank is entangled is strictly between 0 and 1.

As seen above, Arveson's results show that the probability of entanglement of a state ρ depends on the rank of the state compared to the rank of its restriction ω to M_n. He considers the set $E(\omega)$ of states that restrict to ω. He filters those states by rank, with $E^r(\omega)$ being the states in $E(\omega)$ of rank $\leq r$. He shows that almost every state in $E^r(\omega)$ has rank r. Thus in $E(\omega)$ almost all states have the maximum possible rank $m \cdot \operatorname{rank} \omega$. Hence in determining the probability of entanglement for states of rank r the relevant context is $E^r(\omega)$, not $E(\omega)$.

Arveson's central idea is to use the "noncommutative sphere" $V^r(n,m)$ as a parameter space. This is the set of r-tuples $v = (v_1, \ldots, v_r)$ of complex $m \times n$ matrices satisfying

$$v_1^* v_1 + v_2^* v_2 + \cdots + v_r^* v_r = 1_n. \tag{1}$$

For each r this is a real-analytic manifold, with a natural transitive action of a compact group. There is a unique invariant probability measure on $V^r(n,m)$.

Arveson first parameterizes UCP maps. A UCP map $\Phi : M_m \to M_n$ is a completely positive map that is unital, i.e. $\Phi(I_m) = I_n$. UCP maps are the dual maps of quantum channels with respect to the duality given by the Hilbert-Schmidt inner product $\langle A, B \rangle = \operatorname{tr}(B^* A)$. Thus parameterizing UCP maps is equivalent to parameterizing quantum channels.

For each $v \in V^r(n,m)$ he defines the UCP map $\Phi_v : M_m \to M_n$ by

$$\Phi_v(x) = \sum_{k=1}^{r} v_k^* x v_k.$$

Every UCP map Φ arises in this way for some r. The matrices v_k can be chosen to be linearly independent, and in that case r is unique and is called the rank of Φ. The probability measure on $V^r(n,m)$ induces a probability measure on the set of UCP maps of rank $\leq r$.

Next he defines a map from UCP maps of rank $\leq r$ onto $E^r(\omega)$. Let $\xi \in \mathbb{C}^n \otimes \mathbb{C}^n$ be a unit vector such that the vector state ω_ξ on $M_n \otimes M_n$

extends ω. Define a state ρ_Φ on $M_m \otimes M_n$ by

$$\rho_\Phi(a \otimes b) = \langle (\Phi(a) \otimes b)\xi, \xi \rangle. \tag{2}$$

The map $\Phi \mapsto \rho_\Phi$ is a homeomorphism from the set of UCP maps of rank $\leq r$ onto $E^r(\omega)$. Composing this with the map $v \mapsto \Phi_v$ gives a parameterization of $E^r(\omega)$, and then the probability measure on $V^r(n, m)$ induces a measure on $E^r(\omega)$.

The map $\Phi \mapsto \rho_\Phi$ generalizes other well known correspondences of completely positive maps and positive linear functionals, cf. [5, 6, 11, 13]. Perhaps the earliest use of such a correspondence was by Arveson [1] in his work on extensions of completely positive maps.

Arveson's main tools in proving his results are showing that particular sets of parameters are open (hence have positive measure), or are a proper subvariety of $V^r(n, m)$ (hence have measure zero). Arveson identifies parameters associated with separable or entangled states directly in terms of a condition on the parameters. He also defines a "wedge invariant" on $V^r(n, m)$ which provides a new necessary condition for separability, quite different from previously known conditions.

Next we discuss Arveson's paper "Quantum channels that preserve separability". Arveson calls a vector $\xi \in \mathbb{C}^n \otimes \mathbb{C}^n$ *highly entangled* if ω_ξ restricted to M_n has rank n, or equivalently if $\xi = \sum_{i=1}^n \eta_i \otimes \nu_i$, where $\eta_1, \ldots \eta_n$ are nonzero and orthogonal, and $\nu_1, \ldots, \nu_p$ are nonzero and orthogonal. Arveson then says a UCP map Φ is *entanglement preserving* if the adjoint map $(\Phi \otimes I)'$ maps all highly entangled vector states to entangled states.

On the other hand, in quantum information theory a UCP map Φ is said to be *entanglement breaking* if the map $(\Phi \otimes I)'$ takes all states to separable states. Arveson proves the interesting dichotomy that every UCP map is either entanglement preserving or entanglement breaking.

A UCP map Φ is entanglement preserving iff the corresponding state ρ_Φ is entangled. This is used to carry over results in [2] on probability of states being entangled to statements about the probability of UCP maps being entanglement preserving. For example, Arveson shows that UCP maps of rank $\leq n/2$ are almost surely entanglement preserving, but that this is not the case for rank mn.

The paper [3] finishes with some results on the likelihood of a UCP map being extremal (i.e., an extreme point of the set of UCP maps). Arveson shows that the set of extremal UCP maps of rank r is a relatively open and dense set of full measure in all UCP maps of rank r, and that there are no extremal UCP maps of rank $> n$. This uses Choi's characterization of extremal UCP maps [5].

We turn now to Arveson's third paper "Maximal vectors in Hilbert space and quantum entanglement" [4]. In quantum information, entanglement is considered a resource, and some entangled states are viewed as possessing more entanglement than others. However, there is no agreed upon definition of the amount of entanglement other than for bipartite vector states.

The context here is a Hilbert space H which is a tensor product of Hilbert spaces $H_1, \dots, H_N$, so $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N$. Letting $n_k = \dim H_k$, we can arrange $n_1 \leq n_2 \ldots \leq n_N$, and Arveson allows H_N to be infinite dimensional. To cover both the cases where H is finite or infinite dimensional, the role of matrix algebras M_n is played by $B(H)$, the bounded operators on H.

If $N = 2$ (the "bipartite" case), a unit vector is separable if it is a product vector $\xi = \eta_1 \otimes \eta_2$, and otherwise is entangled. Various criteria for measuring the amount of entanglement coincide in this case, and there is general agreement that a unit vector is *maximally entangled* precisely if it can be written in the form

$$\frac{1}{\sqrt{n_1}} \sum_i \xi_i \otimes \eta_i \tag{3}$$

where $\xi_1, \cdots, \xi_{n_1}$ is an orthonormal basis of H_1 and $\eta_1, \cdots, \eta_{n_1}$ are orthonormal in H_2.

The situation is more complicated for pure states when $N > 2$, and for general states. Arveson begins by defining a *decomposable vector* to be one of the form $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N$ where $\xi_i \in H_i$ for $1 \leq i \leq N$. Let V be the set of decomposable unit vectors. Then he defines a *maximal vector* to be one whose distance from V is maximal. He shows that the maximal vectors are the same as the maximally entangled vectors (those of the form (3)) in the bipartite case, and describes both maximal vectors and maximally entangled states in many cases, as we now discuss.

Arveson starts by working with an arbitrary set V of unit vectors in a Hilbert space H. The set V is assumed to be closed under multiplication by scalars of modulus 1, and to have a span that is dense in H. If V isn't closed, one replaces V by its closure. Let K be the closed convex hull of V. He defines the inner radius $r(V)$ to be the radius of the largest closed ball around the origin contained in K. When $r(V) > 0$, then K is the closed unit ball of a unique norm $\|\cdot\|^V$ on H. The requirement $r(V) > 0$ is automatic when H is finite dimensional, and is assumed when $\dim H = \infty$.

Arveson shows the unit vectors where $\|\cdot\|^V$ achieves its minimum are precisely the points in V, i.e., the decomposable vectors. The maximum on the unit sphere is achieved at points where $\|x\|^V = 1/r(V)$, and he proves that these are the maximal vectors, i.e., the points at the maximum distance (in the usual Hilbert norm) from V.

To gain some intuition about Arveson's approach, consider a toy example. Let $H = \mathbb{R}^2$, and let V consists of the four points where the lines $y = \pm x$ meet the unit circle. The convex hull K of V is a square. and the maximal vectors are the four points where the axes meet the unit circle. These are the points ξ whose norm $\|x\|^V$ is maximal and equals $1/r(V)$.

To define and determine maximal states, Arveson proceeds in similar fashion, beginning with the functionals $\omega_{\xi,\eta}$ defined by $\omega_{\xi,\eta}(A) = \langle A\xi, \eta\rangle$ for $\xi, \eta \in V$. (For simplicity in our description of the norm, we assume $\dim H < \infty$.) The closed convex hull $\mathcal{B}$ of V is the closed unit ball of a norm E. The states on which E achieves its minimum value 1 are the convex combinations of vector states ω_ξ for $\xi \in V$: a generalization of separable states. Arveson calls $E(\rho)$ the *generalized entanglement* of ρ, and a state is defined to be *maximally entangled* if E achieves its maximum value $r(V)^{-2}$ at ρ.

Next these abstract results are applied to the concrete case of interest. Arveson identifies the norms $\|\cdot\|^V$ on $H = H_1 \otimes \cdots \otimes H_N$ and the norm E on $B(H) = B(H_1) \otimes \cdots \otimes B(H_N)$ as the projective norms on the tensor products (i.e., the greatest cross norms). (The norm E was used in the bipartite ($N = 2$) case by Rudolph [9] to identify separable states as those with projective norm 1, so Arveson's results both provide a motivation for the role of the projective norm, and generalize Rudolph's separability criterion to cases where $N > 2$.)

Arveson is able to compute $r(V)$ and hence describe the maximal vectors explicitly with an assumption on the dimensions of the Hilbert spaces. The requirement is

$$\dim(H_1 \otimes H_2 \otimes \cdots \otimes H_{N-1}) \leq \dim H_N. \tag{4}$$

He shows that maximal vectors are the vectors whose restriction to $B(H_1 \otimes \cdots \otimes H_{N-1}))$ is the tracial state. In the finite dimensional case, vectors with such restrictions exist iff (4) holds. Assuming (4), he also proves the surprising result that maximal vectors are the vectors that are maximal with respect to the bipartite factorization $(H_1 \otimes \cdots \otimes H_{N-1}) \otimes H_N$, and hence have the explicit description (3) with $\xi_i \in H_1 \otimes \cdots \otimes H_{N-1}$ and $\eta_i \in H_N$. (This also shows that Arveson's notion of maximal vectors coincides with the usual notion of maximally entangled vectors when $N = 2$.)

Since his measure of entanglement is a norm, Arveson is able to show that if a maximally entangled state is a convex combination of other states, each of the latter must also be maximally entangled, and in particular every maximally entangled state is a convex combination of maximally entangled vector states. It also follows that every vector in the range of a maximally entangled state is maximally entangled. Thus the subspaces occurring in this way have the remarkable property that all of their vectors are maximally entangled. It is not apparent that such subspaces of dim > 1 even exist, but Parasarathy [8] has given examples, which he calls perfectly entangled subspaces.

One strength of Arveson's paper is that the results hold for many infinite dimensional Hilbert spaces. On the other hand, cases like $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ don't satisfy the dimension requirement (4), so as Arveson points out, there is research remaining to be done.

I'll finish this summary with a few personal impressions of Arveson. If I was forced to pick a single adjective, it would be "imaginative". In many of the talks I heard him give, he introduced fascinating new concepts, or approached current problems from a surprising direction. In addition to his creativity and technical power, he was a great expositor and speaker. His work on entanglement illustrates all of this.

Finally, I would like to thank Erik Alfsen for many helpful comments on this essay.

References

[1] W. Arveson, Subalgebras of C*-algebras, *Acta Math.* **123** (1969) 141–224.
[2] W. Arveson, The probability of entanglement, *Commun. Math. Phys.* **286** (2009) 283–312.
[3] W. Arveson, Quantum channels that preserve entanglement, *Math. Ann.***343** (2009) 757–771.
[4] W. Arveson, Maximal vectors in Hilbert space and quantum entanglement, *J. Func. Analysis* **256** (2009) 1476–1510.
[5] M.-D. Choi, Completely positive maps on complex matrices, *Lin. Alg. Appl.* **10** (1975) 285–290.
[6] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Reports on Mathematical Physics **3** (1972) 275–278.
[7] Kraus, K., General state changes in quantum theory, *Ann. Phys.* **64**, (1971) 311–335.
[8] K. Parthasarathy, On the maximal dimension of a completely entangled subspace for finite level quantum systems, *Proceedings Mathematical Sciences* **114** (2004) 365–374.
[9] O. Rudolph, A separability criterion for density operators, *J. Phys. A: Math. Gen.* **33** (2000) 3951–55.
[10] M.B. Ruskai, E. Werner, Bipartite states of low rank are almost surely entangled, J. Phys. A. Math. Theor. **42** (2009)
[11] E. Størmer, Separable states and positive maps, *J. Func. Analysis* **254** (2008) 2303–2312.

[12] W. F. Stinespring, Positive functions on C*-algebras, *Proc. Amer. Math. Soc.* **6** (1955) 211–216.

[13] E. Størmer, *Positive linear maps of operator algebras*, Springer 2013.

E-mail address: `fshultz@wellesley.edu`

MATHEMATICS DEPARTMENT, WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS 02481, USA