DIMENSION GROUPS FOR INTERVAL MAPS II: THE TRANSITIVE CASE

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Abstract. Any continuous, transitive, piecewise monotonic map is determined up to a binary choice by its dimension module with the associated finite sequence of generators. The dimension module by itself determines the topological entropy of any transitive piecewise monotonic map, and determines any transitive unimodal map up to conjugacy. For a transitive piecewise monotonic map which is not essentially injective, the associated dimension group is a direct sum of simple dimension groups, each with a unique state.

1. Introduction

Motivated by Elliott's [11] classification of AF-algebras by dimension groups, Krieger [20] gave a dynamical definition of a dimension group for ample groups of homeomorphisms on zero dimensional compact metric spaces in terms of an equivalence relation on the compact open subsets, and used this to associate dimension groups with (two-sided) shifts of finite type. He showed that one of these dimension groups (together with an associated automorphism) is a complete invariant for shift equivalence of topologically mixing shifts of finite type.

The current paper is part of a program to define and investigate dimension groups for piecewise monotonic maps of the unit interval, with the goal of obtaining interesting invariants for such maps. We will see that there are good analogues for interval maps of several well known results for shifts of finite type.

In [32], a dimension group is defined for piecewise monotonic maps of the unit interval. The basic procedure is to associate with the given piecewise monotonic map $\tau : I \to I$ a closely related local homeomorphism $\sigma : X \to X$, where X is formed by disconnecting the interval at appropriate points. Then, following Boyle, Fiebig, and Fiebig [5], the dimension group $DG(\tau)$ is defined to be $C(X, \mathbb{Z})$, modulo the equivalence relation given by $f \sim g$ if $\mathcal{L}^n f = \mathcal{L}^n g$ for some $n \geq 0$, where $\mathcal{L} = \mathcal{L}_\sigma$ is the transfer operator

$$(\mathcal{L}_\sigma f)(x) = \sum_{\sigma y = x} f(y).$$

This dimension group comes equipped with a natural injective endomorphism $\mathcal{L}_* : [f] \mapsto [\mathcal{L}f]$, which is an automorphism if τ is surjective. The triple $(DG(\tau), DG(\tau)^+, \mathcal{L}_*)$ is called the dimension triple associated with τ. In the current paper, we strengthen the results in [32] by restricting consideration to transitive maps.

For a transitive one-sided shift of finite type (X_A, σ_A), there is a period p, and a partition of X into p clopen pieces permuted by σ, such that σ^p is topologically

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mixing on each. The associated dimension group for σ_A will be a direct sum of simple dimension groups.

We establish similar results for transitive piecewise monotonic maps. If τ is transitive, and the associated local homeomorphism σ is not a homeomorphism, then there exists a positive integer N and a decomposition of X into clopen sets $X_1, \ldots, X_N$, permuted by σ, such that σ^N is topologically exact on each X_i (Theorem 4.5). (A map $\sigma : X_i \to X_i$ is topologically exact if for every non-empty open set $V \subset X_i$, there exists $n \geq 0$ such that $\sigma^n(V) = X_i$.) The dimension group $DG(\tau)$ will be a direct sum of simple dimension groups (Theorem 5.3). (Ironically, the case where σ is bijective is the bad case: such a decomposition need not exist for such maps.)

The same kind of decomposition holds for τ, except that the clopen sets are replaced by finite unions of closed intervals, which may overlap at endpoints. This result for τ (with topologically mixing instead of topologically exact) can be obtained from the decomposition of the non-wandering sets of piecewise monotonic maps due to Hofbauer [14, Thm. 2]. General results on "regular periodic decompositions" of (continuous) transitive maps can be found in [2].

The order on a dimension group is an important part of its structure, and for transitive maps we can give an effective description of that order. For a simple dimension group, the order is determined by the states, i.e., the normalized order preserving homomorphisms into $\mathbb{R}$, cf., e.g., [10]. If $\tau : I \to I$ is topologically exact, the dimension group $DG(\tau)$ is simple, with a unique state (Corollary 5.5).

More generally, when τ is transitive, there is a state that is scaled by $\mathcal{L}_*$ by a factor $s \geq 1$. This state is given by a measure on I that is scaled by τ by the factor s. If τ is essentially injective (i.e., there are no intervals J_1, J_2 with $\tau(J_1) = \tau(J_2)$), or equivalently, if the associated local homeomorphism σ is not a homeomorphism, then transitivity of τ implies that the state scaled by $\mathcal{L}_*$ is unique, given by the unique measure on I scaled by τ, with the scaling factor $s = \exp h_\tau$, where h_τ is the topological entropy (Theorem 5.3). It follows that for transitive maps, the dimension triple determines the topological entropy of the map. Transitive unimodal maps are determined by their topological entropy, and thus two such maps are determined up to conjugacy by their dimension triples.

If τ is surjective, then $\mathcal{L}_*$ is an automorphism of $DG(\tau)$, so the latter can be viewed as a $\mathbb{Z}[t, t^{-1}]$ module, called the dimension module. In [32], a canonical finite sequence of generators for this module is identified. The dimension module with its canonical sequence of generators, determines a continuous transitive map up to a binary choice, namely, whether the function increases or decreases on the first interval of monotonicity (Theorem 6.2).

For a two-sided irreducible shift of finite type, Krieger [20] showed that the measure of maximal entropy gives an imbedding of the dimension group modulo the subgroup of infinitesimals into $\mathbb{R}^n$, and that the range of this imbedding on the clopen subsets is an easily computable invariant, powerful enough to distinguish mixing shifts with the same zeta function. This quotient group has also proved to be a valuable invariant for minimal homeomorphisms of Cantor sets, cf. [12].

If $\tau : I \to I$ is a transitive piecewise monotonic map, the infinitesimal elements in $DG(\tau)$ are precisely those killed by all states. We show that the unique measure on the unit interval I scaled by τ induces an isomorphism from $DG(\tau)/DG(\tau)_{\mathrm{inf}}$

into $\mathbb{R}^n$ (Proposition 5.7). This provides an alternative invariant, which is easy to compute in terms of the measure scaled by τ.

For a primitive matrix A, the Perron-Frobenius theorem establishes existence of a unique eigenvector for the maximum eigenvalue λ, and convergence of powers of $\lambda^{-1}A$. In the current paper, the role of $\lambda^{-1}A$ is played by $s^{-1}\mathcal{L}_{\sigma}*$, which is the Perron-Frobenius operator. Convergence of powers of this operator on the space $\mathcal{B}V$ of functions in L^1 of bounded variation has been thoroughly investigated, e.g. [30, 15, 31, 33, 34, 29]. An exposition can be found in the book [4], and we mainly follow [31]. For our purposes, it is important to have not just convergence in the $\mathcal{B}V$ norm, but uniform convergence of continuous functions. Since σ is a local homeomorphism, then $C(X)$ is invariant under $\mathcal{L}_{\sigma}$, while $C(I)$ is not invariant under $\mathcal{L}_{\tau}$. Thus we work with σ rather than with τ, and establish the necessary results on continuity and uniform convergence in an appendix. Similar uniform convergence results for the case of β-transformations were established by Walters [33]. These results are the key technical tools used to establish the decomposition of transitive maps into topologically exact pieces, and the results on uniqueness of scaled states described above.

We now summarize this paper. We begin with background on dimension groups of interval maps, mostly taken from [32]. Then scaling measures are introduced, which are measures μ on I such that there is $s > 0$ such that $\mu(\tau(E)) = s\mu(E)$ for all Borel sets E on which τ is injective. These are a special case of conformal measures ([8]), and are closely related to conjugacies of piecewise monotonic maps with uniformly piecewise linear maps, i.e., piecewise linear maps with slopes $\pm s$. (By a result of Parry [25], every transitive piecewise monotonic map is conjugate to a uniformly piecewise linear map, cf. Corollary 4.4.) This is mostly standard material, adapted to the current context. For our purpose, the key fact is that such measures induce states on the dimension group.

We next prove the finite decomposition of non-injective transitive maps into topologically exact maps, and the uniqueness of the state scaled by $\mathcal{L}_*$. A similar uniqueness result for topologically exact positively expansive maps was proven by Renault [27], who used a somewhat different dimension group, based on inductive limits of $C(X)$ rather than $C(X, \mathbb{Z})$.

We finish by computing the dimension triple (or the quotient by the subgroup of infinitesimals) for several families of uniformly piecewise linear maps, including tent maps and β-transformations. In the latter case, Katayama, and Watatani [17] have associated C*-algebras F_β^∞ and O_β with the β-transformation, and $K_0(F_\beta^\infty)$ is a dimension group. We show that $K_0(F_\beta^\infty)$ is isomorphic to $DG(\tau)$ as a group (and is order isomorphic if the orbit of 1 is eventually periodic.)

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2. Background

We first review the construction in [32], which to each piecewise monotonic map $\tau : [0, 1] \to [0, 1]$ associates a local homeomorphism $\sigma : X \to X$ on a compact set $X \subset \mathbb{R}$, where X is constructed by disconnecting $[0, 1]$ at certain points. (For references to related work of other authors, see [32].)

Definition 2.1. Let $I = [0,1]$. A map $\tau : I \to I$ is *piecewise monotonic* if there are points $0 = a_0 < a_1 < \ldots < a_n = 1$ such that $\tau|(a_{i-1}, a_i)$ is continuous and strictly monotonic for $1 \le i \le n$. We will assume the sequence $a_0, a_1, \ldots, a_n$ is chosen so that no interval (a_{i-1}, a_i) is contained in a larger open interval on which τ is continuous and strictly monotonic. The sequence of points $0 = a_0 < a_1 < \cdots < a_n = 1$ is the *partition associated with* τ, and the intervals $\{(a_{i-1}, a_i) \mid 1 \le i \le n\}$ are called the *intervals of monotonicity for* τ. Note that for $1 \le i \le n$, the map $\tau|(a_{i-1}, a_i)$ extends uniquely to a strictly monotonic continuous map $\tau_i : [a_{i-1}, a_i] \to I$, which will be a homeomorphism onto its image.

If τ is not continuous, we will ignore the actual values of τ at the partition points, and instead view τ as being multivalued, with the values at a_i (for $1 \le i \le n-1$) being the values given by left and right limits, i.e. the values of τ at a_i are $\tau_i(a_i)$ and $\tau_{i+1}(a_i)$. We define a (possibly multivalued) function $\widehat{\tau}$ on I by setting $\widehat{\tau}(x)$ to be the set of left and right limits of τ at x. At points where τ is continuous, $\widehat{\tau}(x) = \{\tau(x)\}$, and we identify $\widehat{\tau}(x)$ with $\tau(x)$. Thus for $A \subset I$, $\widehat{\tau}(A) = \bigcup_i \tau_i(A \cap [a_{i-1}, a_i])$, and $\widehat{\tau}^{-1}(A) = \bigcup_i \{x \in [a_{i-1}, a_i] \mid \tau_i(x) \in A\}$.

If $x \in I$, the *generalized orbit* of x is the smallest subset of I containing x and forward and backward invariant with respect to $\widehat{\tau}$. Let I_1 be the union of the generalized orbits of $a_0, a_1, \ldots, a_n$, and let $I_0 = I \setminus I_1$.

Definition 2.2. Let $I = [0,1]$, and let I_0, I_1 be as above. The *disconnection of* I *at points in* I_1 is the totally ordered set X which consists of a copy of I with the usual ordering, but with each point $x \in I_1 \setminus \{0,1\}$ replaced by two points $x^- < x^+$. We equip X with the order topology, and define the *collapse map* $\pi : X \to I$ by $\pi(x^\pm) = x$ for $x \in I_1$, and $\pi(x) = x$ for $x \in I_0$. We write $X_1 = \pi^{-1}(I_1)$, and $X_0 = \pi^{-1}(I_0) = X \setminus X_1$.

Notation. For any pair $a, b \in X$, we write $[a,b]_X$ for the order interval $\{x \in X \mid a \le x \le b\}$. If $b_1, b_2 \in I_1$ with $b_1 < b_2$, then $I(b_1, b_2)$ is the order interval $[b_1^+, b_2^-]_X$. If $b_1 > b_2$, then we define $I(b_1, b_2) = I(b_2, b_1)$, and if $b_1 = b_2$, then we set $I(b_1, b_2) = \emptyset$.

Proposition 2.3. *(*[32, Prop. 2.2]*) Let X be the disconnection of I at points in I_1, and $\pi : X \to I$ the collapse map. Then X is homeomorphic to a compact subset of $\mathbb{R}$, and*

(i) *π is continuous and order preserving.*
(ii) *I_0 is dense in I, and X_0 is dense in X.*
(iii) *$\pi|_{X_0}$ is a homeomorphism from X_0 onto I_0.*
(iv) *X has no isolated points.*
(v) *If $a, b \in I_1$, then $I(a,b)$ is clopen in X, and every clopen subset of X is a finite disjoint union of such order intervals.*

Proposition 2.4. *(*[32, Thm. 2.3]*) Let $\tau : I \to I$ be a piecewise monotonic map, with associated partition $a_0 < a_1 < \cdots < a_n$, and let (X, π) be as described in Proposition 2.3.*

(i) *There is a unique continuous map $\sigma : X \to X$ such that $\pi \circ \sigma = \tau \circ \pi$ on X_0.*
(ii) *The sets X_0 and $X_1 = X \setminus X_0$ are forward and backward invariant with respect to σ.*
(iii) *π is a conjugacy from $\sigma|X_0$ onto $\tau|I_0$.*

(iv) *The sets* $J_1 = I(a_0, a_1), \ldots, J_n = I(a_{n-1}, a_n)$ *are a partition of* X *into clopen sets such that for* $1 \le i \le n$, $\pi(J_i) = [a_{i-1}, a_i]$, *and* $\sigma|J_i$ *is a homeomorphism from* J_i *onto the clopen set* $\sigma(J_i)$.
(v) *For* $1 \le i \le n$, $\pi \circ \sigma = \tau_i \circ \pi$ *on* J_i.

Note that if τ is continuous, by (iii) and density of I_0 in I and X_0 in X, π will be a semi-conjugacy from (X, σ) onto (I, τ).

If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the map in Proposition 2.4, then σ will be a local homeomorphism by Proposition 2.4 (iv). We will call σ the *local homeomorphism associated with* τ. Property (iii) of Proposition 2.4 can be used to show that σ and τ share many properties. Before being more explicit, we review some terminology.

Definition 2.5. If X is any topological space, and $f : X \to X$ is a continuous map, then f is *transitive* if for each pair U, V of non-empty open sets, there exists $n \ge 0$ such that $f^n(U) \cap V \neq \emptyset$. We say f is *strongly transitive* if for every non-empty open set U, there exists n such that $\cup_{k=0}^{n} f^k(U) = X$, and f is *topologically exact* if for every non-empty open set U, there exists n such that $f^n(U) = X$.

The notion of strong transitivity appears in Parry's paper [25], and a discussion of topological exactness for interval maps can be found in the book of Preston, cf., e.g., [26, pp. 6, 27].

Definition 2.6. If $\tau : I \to I$ is piecewise monotonic, we view τ as undefined at the set C of endpoints of intervals of monotonicity, and say τ is *transitive* if for each pair U, V of non-empty open sets, there exists $n \ge 0$ such that $\tau^n(U) \cap V \neq \emptyset$. We say τ is *strongly transitive* if for every non-empty open set U, there exists n such that $\cup_{k=0}^{n} \widehat{\tau}^k(U) = I$. (Recall that $\widehat{\tau}$ denotes the (possibly multivalued) function whose value at each point x is given by the left and right hand limits of τ at x.) The map τ is *topologically exact* if for every non-empty open set U, there exists n such that $\widehat{\tau}^n(U) = I$. (See Lemma 8.1, Example 9.3, and Lemma 10.1 for examples of topologically exact maps.)

If $\tau : I \to I$ is continuous and piecewise monotonic, both Definitions 2.5 and 2.6 are applicable, and are consistent. Transitivity is equivalent to the existence of a dense orbit.

The full two sided n-shift is an example of a map that is transitive, but not strongly transitive. (The complement of a fixed point is an invariant open set.) However, for piecewise monotonic maps, these notions are equivalent.

Proposition 2.7. *Let* $\tau : I \to I$ *be piecewise monotonic, and* $\sigma : X \to X$ *the associated local homeomorphism.*

(i) σ *is surjective iff* τ *is surjective.*
(ii) σ *is strongly transitive iff* τ *is strongly transitive iff* τ *is transitive iff* σ *is transitive.*
(iii) σ *is topologically exact iff* τ *is topologically exact.*

Proof. [32, Lemma 4.2, Prop. 2.9, Lemma 5.2]. □

Now we turn to comparing the topological entropies of τ and σ. For a continuous map f on a compact metric space X, we denote the (topological) entropy of f by h_f. In the special case where (X, f) is a subshift, then we have the formula

$$h_f = \lim_{n\to\infty} \tfrac{1}{n} \ln c_n(f), \tag{1}$$

where $c_n(f)$ is the number of cylinders in X of length n, cf. [21, ex. 6.3.4].

If $\tau : I \to I$ is piecewise monotonic, let $a_0 < a_1 < \ldots < a_q$ be the partition associated with τ, and let $A_j = (a_{j-1}, a_j)$ for $1 \le j \le q$. Let

$$\Sigma_\tau = \left\{ (s_i)_{i=0}^{\infty} \in \prod_{0}^{\infty} \{1, \ldots, q\} \mid \bigcap_{i=0}^{n} \tau^{-i}(A_{s_i}) \neq \emptyset \text{ for all } n \geq 0 \right\}. \tag{2}$$

Then Σ_τ is compact (for the product topology), and invariant under the shift map. The shift map on Σ_τ is called the symbolic dynamics of τ. If τ is continuous as well as piecewise monotonic, then the topological entropy of τ is the same as the topological entropy of the symbolic dynamics of τ, cf. [35, Prop. 2.1] or [24]. If τ is piecewise monotonic, but is not continuous everywhere, then we define h_τ to be the topological entropy of the symbolic dynamics of τ.

Proposition 2.8. *If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, then $h_\tau = h_\sigma$.*

Proof. It is readily seen that the symbolic dynamics of τ and of σ coincide (where the symbolic dynamics of σ is taken with respect to the partition $J_1, \ldots, J_n$ in Theorem 2.4.) As remarked above, by [35, Prop. 2.1] the topological entropy of τ is the same as that of its symbolic dynamics. The proof of [35, Prop. 2.1] is given for a continuous piecewise monotonic map $\tau : I \to I$, but applies without change to $\sigma : X \to X$. Thus σ and its symbolic dynamics have the same topological entropy, and so $h_\tau = h_\sigma$. □

Definition 2.9. Let X be a compact Hausdorff space and $\sigma : X \to X$ any map such that all fibers $\sigma^{-1}(x)$ are finite. Then for any $f : X \to \mathbb{R}$, we define $\mathcal{L}_\sigma f$ by

$$(\mathcal{L}_\sigma f)(x) = \sum_{\sigma y = x} f(y),$$

and call $\mathcal{L}_\sigma$ the *transfer map*. We will write $\mathcal{L}$ in place of $\mathcal{L}_\sigma$ when the meaning is clear from the context.

If $\sigma : X \to X$ is a local homeomorphism, and $f : X \to \mathbb{R}$ is continuous, then $\mathcal{L}_\sigma f$ will be continuous (see the remark after Lemma 3.2), so $\mathcal{L}_\sigma$ maps $C(X)$ into $C(X)$, and $C(X, \mathbb{Z})$ into $C(X, \mathbb{Z})$.

If X is a compact metric space, a map $\sigma : X \to X$ is a *piecewise homeomorphism* if σ is continuous and open, and X admits a finite partition into clopen sets $X_1, X_2, \ldots, X_n$ such that σ is a homeomorphism from X_i onto $\sigma(X_i)$ for $i = 1, \ldots, n$. Any local homeomorphism on a zero dimensional compact metric space will be a piecewise homeomorphism, and if $\tau : I \to I$ is piecewise monotonic, then the associated local homeomorphism $\sigma : X \to X$ will be a piecewise homeomorphism (Proposition 2.4).

Definition 2.10. Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. For $f, g \in C(X, \mathbb{Z})$, define $f \sim g$ if there exists $n \geq 0$ such that $\mathcal{L}^n f = \mathcal{L}^n g$. Then G_σ is the ordered abelian group whose elements are equivalence classes $[f]$, with addition $[f] + [g] = [f + g]$, and with order given by $[f] \geq 0$ if $\mathcal{L}^n f \geq 0$ for some n. If $\tau : I \to I$ is piecewise monotonic, with associated local homeomorphism $\sigma : X \to X$, then $DG(\tau)$ is defined to be G_σ.

The ordered groups G_σ and $DG(\tau)$ are dimension groups [32, Cor. 3.12 and Def. 3.13], i.e., are inductive limits of a sequence of groups of the form $\mathbb{Z}^{n_k}$. See [10] or [13] for background on dimension groups.

Definition 2.11. $\mathcal{L}_* : DG(\tau) \to DG(\tau)$ is defined by $\mathcal{L}_*[f] = [\mathcal{L}f]$.

This is an injective homomorphism, and is bipositive, i.e., $[f] \ge 0$ iff $\mathcal{L}_*[f] \ge 0$. If τ is surjective, then $\mathcal{L}_*$ is surjective, and thus is an automorphism of the dimension group $DG(\tau)$. In that case, we define an action of $\mathbb{Z}[t, t^{-1}]$ on $DG(\tau)$ by

$$(\sum_{-n}^{n} a_i t^i)[f] = (\sum_{-n}^{n} a_i \mathcal{L}_*^i)[f],$$

and view $DG(\tau)$ as a $\mathbb{Z}[t, t^{-1}]$ module. The following describes a set of generators for this module. In the statement of the theorem, we identify order intervals $I(a, b)$ in X with the equivalence class in $DG(\tau)$ of their characteristic functions.

Theorem 2.12. *Let $\tau : I \to I$ be piecewise monotonic and surjective, with associated partition $C = \{a_0, a_1, \dots, a_n\}$. Let*

$$\mathcal{J}_1 = \{\, I(c, d) \mid c,\ d \text{ are adjacent points in } \{\, a_0, a_1, \dots, a_n \,\}\,\}, \tag{3}$$

and let $\mathcal{J}_2$ be the set of intervals corresponding to jumps at partition points, i.e.,

$$\mathcal{J}_2 = \{\, I(\tau_i(a_i), \tau_{i+1}(a_i)) \mid 1 \le i \le n-1 \,\}. \tag{4}$$

Then $DG(\tau)$ is generated as a module by $\mathcal{J}_1 \cup \mathcal{J}_2$.

Proof. [32, Thm. 6.2] □

Corollary 2.13. *If $\tau : I \to I$ is a continuous, surjective piecewise monotonic map with associated partition $\{a_0, a_1, \dots, a_n\}$, then $DG(\tau)$ is generated as a module by $I(a_0, a_1)$, $I(a_1, a_2)$, $\dots$, $I(a_{n-1}, a_n)$.*

Proof. [32, Cor. 6.3] □

3. Scaling measures, and uniformly piecewise linear maps

For simple dimension groups, the order is determined by states, i.e., positive homomorphisms into $\mathbb{R}$, cf. [10]. In the case that τ is transitive, we will see that states are given by scaling measures. In this section, we provide background for such measures, and discuss their connection with uniformly piecewise linear maps.

Let X be a compact metric space and m a probability measure on X (i.e. a positive regular Borel measure with $m(X) = 1$.) If $f \in L^1(X, m)$, we will usually write $m(f)$ instead of $\int f \, dm$.

Definition 3.1. Let X be a compact metric space and m a probability measure on X, and $\sigma : X \to X$ a map that takes Borel sets to Borel sets. Then σ *scales m by a factor s* if $m(\sigma(E)) = s\, m(E)$ for all Borel sets E on which σ is 1–1. (This is a special case of the notion of a *conformal measure*, cf. [8].) The measure m has *full support* if its support is all of X.

Now we will show that scaling measures are just the eigenvectors of the dual of the transfer operator. We write $C(X)$ for the Banach space of real valued continuous functions with the supremum norm. Recall for a local homeomorphism $\sigma : X \to X$, $\mathcal{L}$ is the transfer operator $\mathcal{L}_\sigma$ (Definition 2.9).

The following result, in the more general context of conformal measures, can be found in [8, Prop. 2.2].

Lemma 3.2. *Let $\tau : I \to I$ be piecewise monotonic, and let $\sigma : X \to X$ be the associated local homeomorphism. Let μ be a probability measure on X, and $0 < s \in \mathbb{R}$. The following are equivalent.*

(i) $\mu(\mathcal{L}f) = s\mu(f)$ *for all* $f \in C(X)$.
(ii) σ *scales* μ *by the factor* s.
(iii) $\mathcal{L}$ *is well defined on* $L^1(X, \mu)$, *and* $\mu(\mathcal{L}f) = s\mu(f)$ *for all* $f \in L^1(X, \mu)$.

Proof. (i) $\Rightarrow$ (ii). Let $J_1, \ldots, J_n$ be a partition of X into clopen sets such that σ is injective on each J_i, cf. Proposition 2.4. Let $E_i = \sigma(J_i)$, and let $\psi_i : E_i \to J_i$ be the inverse of $\sigma|J_i$. Let $f \in C(X)$ have support in J_i. Then $\mathcal{L}f$ has support in E_i and is defined on E_i by $\mathcal{L}f = f \circ \psi_i$. Thus

$$s \int_{J_i} f \, d\mu = s\mu(f) = \mu(\mathcal{L}f) = \int_{E_i} f \circ \psi_i \, d\mu = \int_{J_i} f \, d(\psi_i \cdot \mu), \tag{5}$$

where $(\psi_i \cdot \mu)(E) = \mu(\sigma(E))$ for $E \subset J_i$. Since this holds for all $f \in C(J_i)$, it follows that the regular Borel measures $s\mu$ and $\psi_i \cdot \mu$ must coincide on J_i. Thus for $E \subset J_i$ we have $\mu(\sigma(E)) = s\mu(E)$. Therefore $\mu(\sigma(E \cap J_i)) = s\mu(E \cap J_i)$ for all Borel sets E. It follows that $\mu(\sigma(E)) = s\mu(E)$ for all Borel sets E on which σ is injective.

(ii) $\Rightarrow$ (iii). If A is a Borel set and $\mu(A) = 0$, then $\mu(\sigma(A \cap J_i)) = 0$ for all i, so $\mu(\sigma(A)) = 0$. If $f = 0$ a.e., since

$$\{x \mid (\mathcal{L}f)(x) \neq 0\} \subset \sigma(\{x \mid f(x) \neq 0\}),$$

then $\mathcal{L}f = 0$ a.e. Thus $\mathcal{L}$ is well-defined on $L^1(X, \mu)$. Reversing the argument in the first paragraph (but with $f \in L^1(X, \mu)$ in (5)) proves (iii). Trivially (iii) implies (i). □

We note for future reference that the proof of Lemma 3.2 shows that $\mathcal{L}_\sigma$ maps $C(X)$ into $C(X)$.

Recall that a probability measure m on a compact metric space is *non-atomic* if m takes the value 0 on each singleton subset $\{x\}$ of X.

Proposition 3.3. *Let $\tau : I \to I$ be piecewise monotonic, with associated local homeomorphism $\sigma : X \to X$, and collapse map $\pi : X \to I$. Then $\mu \mapsto \mu \circ \pi$ is a 1-1 correspondence of non-atomic probability measures on I of full support, scaled by τ by a factor s, and non-atomic probability measures of full support on X, scaled by σ by the factor s.*

Proof. Since I_1 and X_1 are countable, these sets are killed by any non-atomic measures. The proposition then follows from the fact that the collapse map π is a conjugacy from $\sigma|X_0$ onto $\tau|I_0$ (Proposition 2.4). □

Definition 3.4. A map $f : I \to I$ is *piecewise linear* if f is piecewise monotonic and is linear on the interior of each interval of monotonicity. A map is *uniformly piecewise linear* if it is piecewise linear and the slopes are all $\pm s$ for some $s > 0$.

Example 3.5. If $f : I \to I$ is uniformly piecewise linear with slopes $\pm s$, then Lebesgue measure is scaled by f by a factor s.

This example is canonical, in the sense made precise by Proposition 3.6. This result is implicit in [25].

Proposition 3.6. *A piecewise monotonic map $\tau : I \to I$ is conjugate to a piecewise linear map with slopes $\pm s$ iff there exists a non-atomic probability measure m on I of full support on I, scaled by τ by the factor s. The conjugacy can be chosen to carry m to Lebesgue measure.*

Proof. Suppose such a measure m exists. Define $h : I \to I$ by $h(x) = m((0, x])$ (with $h(0) = 0$). Since m is zero on no open set, then h is strictly increasing. The map h is right continuous by σ-additivity of m, and left continuous because m is non-atomic. Thus h is a continuous injective map of $[0, 1]$ into $[0, 1]$. Since its range includes 0 and 1 and is connected, then h is surjective, so is a homeomorphism.

Now define $f : [0, 1] \to [0, 1]$ by $f = h \circ \tau \circ h^{-1}$. Then $f : [0, 1] \to [0, 1]$ is conjugate to τ. Let $J_1, \ldots, J_n$ be a cover of I by intervals on which τ is monotonic, with only endpoints in common. From the fact that m is scaled by τ by a factor s, it follows that f is linear on the interior of each interval $h(J_i)$, with slope $\pm s$. Thus τ is conjugate to the uniformly piecewise linear map f. By the definition of h, the conjugacy takes the measure m to Lebesgue measure.

The converse follows from Example 3.5. □

The following is well known for continuous piecewise linear maps; in that case it follows at once from [23]. See the discussion preceding Proposition 2.8 for the definition of topological entropy for piecewise monotonic maps that are not continuous.

Proposition 3.7. *If τ is a piecewise linear map with slopes $\pm s$, and $s \geq 1$, then $h_\tau = \ln s$.*

Proof. Let $\sigma : X \to X$ be the local homeomorphism associated with τ, and $\mathcal{L} = \mathcal{L}_\sigma$. Let $R = \lim_n \|\mathcal{L}^n 1\|_\infty^{1/n}$, where $\|\cdot\|_\infty$ denotes the supremum norm. By [29, Thm. 6.1], $R = \exp(h_\sigma)$, and by Proposition 2.8, $h_\sigma = h_\tau$, so to prove the proposition, we need to show that $s = R$.

Let m denote Lebesgue measure. Then m is scaled by τ by the factor s. By Proposition 3.3 there is a unique non-atomic probability measure μ on X, scaled by σ by the factor s, such that $\mu = m \circ \pi$, where $\pi : X \to I$ is the collapse map. We have $s^n = \mu(\mathcal{L}^n 1) \leq \|\mathcal{L}^n 1\|_\infty$, so $s \leq \|L^n 1\|_\infty^{1/n}$. Taking limits, we get $s \leq R$. Suppose that $s < R$. Since $1 \leq s < R$, by [29, Thm. 4.1], there exists a function ϕ of bounded variation which is an eigenvector for $\mathcal{L}$ for the eigenvalue R. If $s > 1$, having such an eigenvector contradicts the spectral radius of $\mathcal{L}$ on the space $\mathcal{B}V$ of functions of bounded variation being s, cf. [31, Thm. 1], so $s = R$. (The space $\mathcal{B}V$ is discussed in more detail in the appendix, cf. Definition A.1.) If $s = 1$, the entropy of τ is 0 (cf. [7]), so again $s = \exp h_\tau$. □

Corollary 3.8. *Let $\tau : I \to I$ be piecewise monotonic. If a non-atomic measure m with full support is scaled by τ by a factor $s \geq 1$, then $s = \exp(h_\tau)$.*

Proof. By Proposition 3.6, τ is conjugate to a piecewise linear map with slopes $\pm s$. By Proposition 3.7, such a map has entropy $\ln s$. □

4. Decomposition of transitive piecewise monotonic maps

In this section we will see that transitive piecewise monotonic maps often can be decomposed into topologically exact pieces.

In [26, Thm. 2.5], it is shown that a transitive, piecewise monotonic, continuous map τ is either topologically exact, or there exists $c \in I$ such that τ exchanges $[0, c]$ and $[c, 1]$, such that τ^2 is topologically exact on each of $[0, c]$ and $[c, 1]$. We will establish a similar finite decomposition for transitive piecewise monotonic maps that are not necessarily continuous. To achieve this decomposition, we need to exclude maps that are "essentially injective".

Definition 4.1. A map $\tau : I \to I$ is *essentially injective* if there are no disjoint intervals J, J' with $\tau(J) = \tau(J')$.

This is equivalent to τ being injective on the complement of the set of endpoints of intervals of monotonicity.

Lemma 4.2. *A piecewise monotonic map $\tau : I \to I$ is essentially injective iff the associated local homeomorphism $\sigma : X \to X$ is injective.*

Proof. [32, Lemma 11.3] □

Proposition 4.3. *If $\tau : I \to I$ is piecewise monotonic and transitive, and $\sigma : X \to X$ is the associated local homeomorphism, then there exists a scaling measure for τ and for σ. Any such scaling measure has full support and is non-atomic, and the scaling factor $s = \exp h_\tau \geq 1$. Here $s > 1$ iff τ is not essentially injective.*

Proof. Note that σ is strongly transitive (Proposition 2.7). Let K be the set of probability measures on X, viewed as positive functionals on $C(X)$. Then K is a w*-compact convex set. Define $T : K \to K$ by $T\nu = (\nu(\mathcal{L}1))^{-1}\nu \circ \mathcal{L}$. (Since σ is transitive, it is surjective, so $\mathcal{L}1 \geq 1$, and thus for all $\nu \in K$, $\nu(L1) \neq 0$). Note that $T\nu$ is again a probability measure, and T is a weak* continuous map. By the Schauder-Tychonov fixed point theorem (cf., e..g, [9, Thm. V.10.5]), T has a fixed point in K, say $T\mu = \mu$. Then $\mu = (\mu(\mathcal{L}1))^{-1}\mu \circ \mathcal{L}$ so $\mu \circ \mathcal{L} = s\mu$, with $s = \mu(\mathcal{L}1)$.

Now let μ be any probability measure scaled by σ by a factor s. Since $\mathcal{L}1 \geq 1$, then $s \geq 1$.

Recall that σ is injective iff τ is essentially injective (Lemma 4.2). For such maps, σ is invertible, and $\mathcal{L}_{\sigma^{-1}} = (\mathcal{L}_\sigma)^{-1}$, so $\mu \circ \mathcal{L}_{\sigma^{-1}} = s^{-1}\mu$. Since σ^{-1} is surjective, $\mathcal{L}_{\sigma^{-1}}1 \geq 1$, so $s^{-1} \geq 1$, which implies $s = 1$.

On the other hand, if σ is not injective on X, suppose that $\sigma(y_1) = \sigma(y_2) = x$. Then $(\mathcal{L}1)(x) \geq 2$, so $\mathcal{L}1 > 1$ on some open set. Thus $\mathcal{L}1 - 1$ is continuous, nonnegative, and strictly positive on an open set. We will see in the next paragraph that μ has full support, so $0 < \mu(L1 - 1) = s - 1$. Thus $s > 1$.

Now we show that μ has full support. Let V be any open subset of X, and suppose that $\mu(V) = 0$. By Proposition 2.4, there is a partition of X into clopen sets on which σ is injective. Thus we can write V as a finite union of open sets on which σ is injective, say $V = \cup V_i$. Then $\mu(V_i) = 0$ for each i, so

$$\mu(\sigma(V)) = \mu(\sigma(\cup V_i) = \mu(\cup\sigma(V_i)) \leq \sum \mu(\sigma(V_i)) = \sum s\mu(V_i) = 0. \tag{6}$$

By the same argument, $\mu(\sigma^k(V)) = 0$ for all $k \geq 0$. Since σ is strongly transitive, a finite number of iterates of V cover X, so we conclude that $\mu(X) = 0$, contrary to μ being a probability measure. Thus $\operatorname{supp}\mu = X$.

Finally, we will show μ is non-atomic. If $s > 1$, then $\mu(\sigma^n(\{x\})) = s^n\mu(\sigma(\{x\})) < \mu(\sigma(X))$ for all n implies $\mu(\sigma(\{x\})) = 0$, so μ is non-atomic. If $s = 1$, then as shown above, σ is injective on X, so it is a homeomorphism. Since σ is strongly transitive, it can have no periodic points. (The complement of a periodic orbit would be an invariant open set.) For any $x \in X$, $x, \sigma(x), \ldots$ will be a sequence of distinct points with the same measure, so all must have measure zero. Thus μ is non-atomic. □

We will see in Corollary 4.6 that when τ is transitive and not essentially injective, then the scaling measure is unique.

For any surjective piecewise monotonic map $\tau : I \to I$, with associated local homeomorphism $\sigma : X \to X$, the proof above shows that τ and σ possess scaling measures, and shows that for homeomorphisms the scaling factor must be 1. Thus when τ is surjective and essentially injective (or equivalently, σ is a homeomorphism), scaling measures are the same as invariant measures.

If τ is strongly transitive, the following corollary is in [25], with an extra aperiodicity assumption if τ is essentially injective.

Corollary 4.4. *(Parry) Let $\tau : I \to I$ be piecewise monotonic and transitive. Then τ is conjugate to a piecewise linear function on I with slope $\pm s$, where $s = \exp(h_\tau)$. Here $s = 1$ iff τ is essentially injective.*

Proof. By Proposition 4.3, there exists a non-atomic scaling measure for τ, with full support, and with scaling factor $s \geq 1$. The existence of the desired conjugacy follows from Proposition 3.6. □

Theorem 4.5. *Let $\tau : I \to I$ be piecewise monotonic and transitive, and not essentially injective, and let $\sigma : X \to X$ be the associated local homeomorphism. There is a unique partition of X into clopen sets $X_1, \ldots, X_N$ such that $\sigma(X_i) = X_{i+1 \bmod N}$, with $\sigma^N|X_i$ topologically exact. Let μ be a measure scaled by σ, with scaling factor s. For each i there exists $\phi_i \in C(X)$ of bounded variation, with $\operatorname{supp}\phi_i = X_i$ and $\phi_i > 0$ on X_i, such that for each $f \in C(X)$ with bounded variation, with $\operatorname{supp} f \subset X_i$,*

$$\lim_k (s^{-N}\mathcal{L}_\sigma^N)^k f = \mu(f)\phi_i \text{ (uniform convergence)}, \tag{7}$$

and such that the functions ϕ_i are cyclically permuted by $(1/s)\mathcal{L}_\sigma$, i.e.,

$$(1/s)\mathcal{L}_\sigma \phi_i = \phi_{i+1 \bmod N}. \tag{8}$$

Proof. By Proposition 4.3, $s > 1$. We will make use of the results in the appendix. By Corollary A.9, there exists a positive integer N and disjoint clopen sets $X_1, \ldots, X_q$ with σ^N exact on each X_i, and with $\sigma(X_i) = X_{\omega(i)}$ for a permutation ω. Since σ is transitive, the orbit of each X_i must include every X_j, so ω must be a cycle, and the union of all X_i must be X. By Theorem A.6 (vi), $q = N$. If necessary, re-index so that $\sigma(X_i) = X_{i+1 \bmod N}$. The existence of $\phi_i \in C(X)$ satisfying (7) follows from Corollary A.7 and Proposition A.8, and (8) is a consequence of Theorem A.6 (vi).

To prove uniqueness of this decomposition of X, suppose $H_1, \ldots, H_p$ is another partition of X into clopen sets which are permuted cyclically by σ, such that σ^p is exact on each H_i. Suppose that $H_i \cap X_j \neq \emptyset$. Then for k sufficiently large, by exactness of σ^p on each H_i and σ^N on each X_j, $\sigma^{kpN}(H_i \cap X_j) = H_i = X_j$. Thus each H_i coincides with some X_j, and uniqueness follows. □

In the case when σ is positively expansive, the decomposition in Theorem 4.5 (into mixing rather than exact pieces) follows from [1, Thm. 3.4.4]. However, the map σ cannot be positively expansive unless the forward orbit under τ of the set C of endpoints of intervals of monotonicity is finite. Indeed, suppose $x \in \widehat{\tau}C \setminus C$ has an infinite orbit which never re-enters C. Then the points $\tau^n(x)^\pm$ are all distinct for $n \geq 0$. Viewing X as a compact subset of $\mathbb{R}$, the sum of the gaps $|\tau^n(x)^+ - \tau^n(x)^-|$ is finite, so $\lim_n |\sigma^n(x^+) - \sigma^n(x^-)| = \lim_n |\tau^n(x)^+ - \tau^n(x)^-| = 0$, which shows that σ is not positively expansive.

Corollary 4.6. *If a piecewise monotonic map $\tau : I \to I$ is transitive, and is not essentially injective, then there is a unique scaling measure for τ, and for the associated local homeomorphism $\sigma : X \to X$.*

Proof. Let μ and ν be scaling measures for σ. By Proposition 4.3, both μ and ν are non-atomic and have full support, and have scaling factors > 1. Let $X_1, X_2, \ldots, X_N$ be the unique decomposition given in Theorem 4.5. Fix an index i, and let ϕ_i satisfy (8), where s is the scaling factor for μ. By (8), $\mathcal{L}_\sigma^N \phi_i = s^N \phi_i$. Applying ν, $\nu(\mathcal{L}_\sigma^N \phi_i) = s^N \nu(\phi_i)$. Since ν has full support, then $\nu(\phi_i) \neq 0$. It follows that the scaling factor for ν is the same as that for μ. (This also follows from Proposition 3.3 and Corollary 3.8.)

Now from (7), for each index i, $\nu(\phi_i) = \mu(\phi_i) = 1$, and if $f \in C(X)$ has bounded variation and has support in X_i, then $\nu(f) = \mu(f)$. Since each clopen subset of X is a finite union of order intervals (Proposition 2.3), the characteristic function of each X_i has bounded variation. If $f \in C(X)$ has bounded variation, then $f\chi_{X_i}$ is continuous, with bounded variation, and has support in X_i. It follows that μ and ν agree on all $f \in C(X)$ of bounded variation. Such functions form a subalgebra of $C(X)$ that is dense by the Stone-Weierstrass theorem, so $\mu = \nu$. □

If τ is essentially injective, the uniqueness result above can fail. For example, there are minimal interval exchange maps with more than one invariant measure, cf. [18] or [19].

The following result also follows from Hofbauer's spectral decomposition for the non-wandering set of piecewise monotonic maps, cf. [14, Thm. 2], with "topologically mixing" in place of "topologically exact".

Corollary 4.7. *Let $\tau : I \to I$ be piecewise monotonic, transitive, and not essentially injective. Then there exist sets $K_1, K_2 \ldots K_N$ such that*

(i) *Each K_i is a finite union of closed intervals.*
(ii) $\cup_{i=1}^N K_i = [0, 1]$.
(iii) *If $i \neq j$, the interiors of K_i and K_j are disjoint.*
(iv) *For each i, τ maps the interior of K_i onto the interior of $K_{i+1 \bmod N}$*
(v) *For each i, τ^N is topologically exact when restricted to K_i.*

The sets $K_1, \ldots, K_N$ are uniquely determined by these properties.

Proof. Let $\sigma : X \to X$ be the associated local homeomorphism, and $\pi : X \to I$ the collapse map. Let $X_1, \ldots, X_N$ be as in Theorem 4.5. Define $K_i = \pi(X_i)$ for $1 \leq i \leq N$. Then (i) follows from the fact that each clopen subset of X is a finite union of order intervals, and that π maps closed order intervals to closed intervals. Since σ^N is exact on $X_1, \ldots, X_n$, then the conjugacy of $\sigma|_{X_0}$ with $\tau|_{I_0}$ implies that τ^N is exact on each K_i. (See [32, proof of Lemma 5.2] for proof of a similar result.) □

Observe that the decompositions in Theorem 4.5 and Corollary 4.7 may not exist if τ is essentially injective. For example, the map $x \mapsto x + \alpha \bmod 1$, where α is irrational, is transitive, but is not topologically exact. There also is no decomposition of some iterate τ^N into exact pieces, since $\tau^N : x \mapsto x + n\alpha \bmod 1$ is transitive, and thus does not leave invariant any proper closed subset of I.

Definition 4.8. Let X be a topological space, and $\sigma : X \to X$ a continuous map. Then σ is *topologically mixing* if for every pair U, V of non-empty open sets, there exists N such that $\sigma^n(U) \cap V \neq \emptyset$ for $n \geq N$. (If $\tau : I \to I$ is piecewise monotonic but discontinuous at some endpoints, we view τ as undefined at the set C of endpoints of intervals of monotonicity.)

As in Proposition 2.7 or [32, Lemma 5.2], one can show that a piecewise monotonic map $\tau : I \to I$ will be topologically mixing iff the associated local homeomorphism $\sigma : X \to X$ has this property.

Every topologically exact map is topologically mixing. The converse is false, e.g., if σ is the two-sided shift on the space Σ_2 of bi-infinite sequences of two symbols, σ is topologically mixing, but is not topologically exact. (The complement of a fixed point is open and invariant.) On the other hand, for continuous piecewise monotonic maps, being topologically mixing and being topologically exact are equivalent, cf. [26, Thm. 2.5]. Furthermore, for piecewise monotonic maps that are not essentially injective, mixing and exactness coincide, as we now show.

Proposition 4.9. *If $\tau : I \to I$ is piecewise monotonic, topologically mixing, and not essentially injective, then τ is topologically exact.*

Proof. Let $\sigma : X \to X$ be the associated local homeomorphism. As observed above, σ will be topologically mixing. Then σ is transitive, so we can partition X into clopen subsets on which some σ^N is exact (Theorem 4.5). But if there is more than one member to the partition, this contradicts σ being topologically mixing. So σ is exact, and therefore so is τ. □

5. Simple dimension groups with unique states

In this section we will see that scaling measures can be used to describe the order on $DG(\tau)$ for transitive maps τ. In particular, scaling measures induce states (defined below), and states in turn determine the order.

Definition 5.1. If G is a dimension group with distinguished order unit e, then a *state* on G is a positive homomorphism ω from G into $\mathbb{R}$ such that $\omega(e) = 1$. The *strict order* on G induced by a state ω is given by $x \leq y$ if either $\omega(x) < \omega(y)$ or $x = y$.

If ω is a state on a dimension group G, and Φ is a positive endomorphism of G, we say ω is *scaled by Φ by the factor s* if $\omega(\Phi(x)) = s\,\omega(x)$ for all $x \in G$.

Proposition 5.2. *Let $\tau : I \to I$ be piecewise monotonic, and let $\sigma : X \to X$ be the associated local homeomorphism. If μ is a probability measure on X scaled by σ by a factor s, then the map $[f] \to \mu(f)$ is a state on $DG(\tau)$. We also denote this state by μ, and μ is scaled by $(\mathcal{L}_\sigma)_*$ by the factor s. If X is totally disconnected, (e.g., if τ is transitive), this gives a 1-1 correspondence of scaled measures and scaled states.*

Proof. Clearly μ is a positive homomorphism from $C(X, \mathbb{Z})$ into $\mathbb{R}$ with value 1 on the function constantly 1. To show μ is well defined on $DG(\tau)$, suppose $f, g \in C(X, \mathbb{Z})$ with $[f] = [g]$. Write $\mathcal{L}$ in place of $\mathcal{L}_\sigma$. Then $\mathcal{L}^n f = \mathcal{L}^n g$ for some $n \geq 0$, so

$$\mu(f) = s^{-n}\mu(\mathcal{L}^n f) = s^{-n}\mu(\mathcal{L}^n g) = \mu(g). \tag{9}$$

Furthermore, if $[f] \geq 0$, then $\mathcal{L}^n f \geq 0$ for some n, so $\mu([f]) = s^{-n}\mu(\mathcal{L}^n f) \geq 0$. Thus $[f] \to \mu(f)$ is a well-defined state on $DG(\tau)$. Furthermore, $\mu(\mathcal{L}_*[g]) = \mu(\mathcal{L}g) = s\,\mu(g) = s\,\mu([g])$, so μ is scaled by $\mathcal{L}_*$ by the factor s.

The map that takes a scaled measure to a scaled state is evidently 1-1. Now assume that X is totally disconnected, and let ω be a state on $DG(\tau)$ scaled by $\mathcal{L}_*$ by a factor s. We will show that ω comes from a scaled measure.

Define $\phi : C(X, \mathbb{Z}) \to \mathbb{R}$ by $\phi(f) = \omega([f])$. Then ϕ is a positive homomorphism from $C(X, \mathbb{Z})$ into R. View $C(X, \mathbb{Z})$ as a subset of $C(X)$, and let $C_{\mathbf{Q}}(X)$ denote the rational linear span of $C(X, \mathbb{Z})$ in $C(X)$. Then ϕ extends uniquely to a positive $\mathbf{Q}$-linear map from $C_{\mathbf{Q}}(X)$ into $\mathbb{R}$. Since $\phi(1) = 1$, by positivity ϕ will satisfy $|\phi(f)| \leq \|f\|_\infty$ for $f \in C_{\mathbf{Q}}(X)$. Since X is totally disconnected, by the Stone-Weierstrass theorem, the (real) linear span of characteristic functions of clopen subsets of X will be dense in $C(X)$. Each such real linear combination of characteristic functions can be uniformly approximated by functions in $C_{\mathbf{Q}}(X)$. It follows that ϕ extends uniquely to a positive linear functional on $C(X)$, also denoted by ϕ.

Let μ be the regular Borel measure such that $\phi(f) = \mu(f)$ for all $f \in C(X)$. Since ω is scaled by $\mathcal{L}_*$ by the factor s, it follows that $\phi : C(X) \to \mathbb{R}$ is scaled by $\mathcal{L}$ by the factor s, and thus by Lemma 3.2, μ is scaled by σ by the factor s. Clearly ω is the scaled state associated with μ, which finishes the proof of the 1-1 correspondence of scaled measures and scaled states. Finally, if τ is transitive, then X is totally disconnected by [32, Prop. 5.8], which completes the proof of the proposition. □

If $\tau : I \to I$ is piecewise monotonic, with associated local homeomorphism $\sigma : X \to X$, and m is a non-atomic measure on I with full support, and μ is the corresponding measure on X (cf. Proposition 3.3), then $\mu(I(a, b)) = m([a, b])$ for all $a, b \in I_1$. Thus we will also refer to the state described in Proposition 5.2 as the state associated with m.

If G, H are dimension groups, $\psi : G \to H$ is a *dimension group isomorphism* if ψ is both an order isomorphism and a group isomorphism.

Theorem 5.3. *Let $\tau : I \to I$ be piecewise monotonic and transitive, and not essentially injective, with associated local homeomorphism $\sigma : X \to X$. Let $X_1, \ldots, X_N$ be the partition of X described in Theorem 4.5, and let μ be the unique measure on X scaled by σ. Then $[f] \mapsto \oplus_i [f\chi_{X_i}]$ is a unital dimension group isomorphism of $DG(\tau)$ onto $\oplus_i G_i$, where $G_i = \{[f] \in DG(\tau) \mid \operatorname{supp} f \subset X_i\}$, with distinguished order unit $[\chi_{X_i}]$. Each G_i is isomorphic to the dimension group associated with $\sigma^N|X_i$ (cf. Definition 2.10), and is a simple dimension group, with a unique state μ_i given by $\mu_i([f]) = |\mu(X_i)|^{-1}\mu(f)$, and with the strict order given by that state. Furthermore, the state on $DG(\tau)$ induced by μ is the unique state scaled by $(\mathcal{L}_\sigma)_*$, and $(\mathcal{L}_\sigma)_*$ maps G_i onto $G_{i+1 \bmod N}$ for each i. The scaling factor is $s = \exp h_\tau$.*

Proof. Write $\mathcal{L}$ for $\mathcal{L}_\sigma$. Since $X_1, \ldots, X_N$ is a partition of X into clopen σ^N-invariant subsets, $(\sigma^N)^{-1}(X_i) = X_i$ for each i. It follows that $[f] \mapsto [f|X_i]$ is a group order isomorphism from G_i onto $G_{\sigma^N|X_i}$. Since σ is a piecewise homeomorphism, then σ^N is also a piecewise homeomorphism, so each G_i is simple by exactness of $\sigma^N|X_i$, cf. [32, Theorem 5.3]. From this the fact that $[f] \mapsto \oplus_i [f|X_i]$ is a dimension group isomorphism is immediate.

Fix an index i, with $1 \leq i \leq N$. We will show that G_i has a unique state given by $\mu|X_i$, and has the strict order given by that state. By Theorem 4.5, there is a continuous function ϕ_i, strictly positive on X_i, such that for each $f \in C(X) \cap \mathcal{B}V$ with support in X_i,

$$\lim_{k\to\infty} s^{-Nk} L^{Nk} f = \mu(f)\phi_i, \tag{10}$$

with uniform convergence. Note that each function in $C(X,\mathbb{Z})$ has bounded variation (since each clopen set in X is a finite union of order intervals, cf. Proposition 2.3). Now suppose $f, g \in C(X,\mathbb{Z})$ have support in X_i, and $\mu(f) < \mu(g)$. Choose $\epsilon > 0$ so that $\epsilon \leq (\mu(g) - \mu(f))\phi_i(x)$ for all $x \in X_i$, so that $\mu(f)\phi_i \leq \mu(g)\phi_i - \epsilon$. Choose n so that

$$\|s^{-n}\mathcal{L}^n f - \mu(f)\phi_i\|_\infty < \frac{\epsilon}{2} \text{ and } \|s^{-n}\mathcal{L}^n g - \mu(g)\phi_i\|_\infty < \frac{\epsilon}{2}. \tag{11}$$

Then

$$s^{-n}\mathcal{L}^n f \leq \mu(f)\phi_i + \frac{\epsilon}{2} \leq \mu(g)\phi_i - \frac{\epsilon}{2} \leq s^{-n}\mathcal{L}^n g. \tag{12}$$

Thus $\mathcal{L}^n f \leq \mathcal{L}^n g$, which implies $[f] \leq [g]$.

Conversely, suppose $[f] \leq [g]$. Choose n so that $\mathcal{L}^n f \leq \mathcal{L}^n g$. Then

$$\mu(f) = \mu(s^{-n}\mathcal{L}^n f) \leq \mu(s^{-n}\mathcal{L}^n g) = \mu(g). \tag{13}$$

If $\mu(f) = \mu(g)$, then μ is zero on the nonnegative continuous function $\mathcal{L}^n g - \mathcal{L}^n f$. Since $\operatorname{supp}\mu = X$, then $\mathcal{L}^n f = \mathcal{L}^n g$, so $[f] = [g]$. Thus either $\mu(f) < \mu(g)$ or else $[f] = [g]$, which completes the proof that the order on G_i is the strict order induced by the state μ_i.

To show there is a unique state on G_i, suppose ω is a state on G_i, and fix $f \in C(X,\mathbb{Z})$ with support in X_i. Let $\epsilon > 0$ be given, and choose a positive integer n such that $2/n < \epsilon$. Now chose integers k and p such that $|\mu_i(f) - (k/p)| < 1/n$. This is equivalent to

$$nk - p < np\,\mu_i(f) < nk + p. \tag{14}$$

Since the ordering on G_i is the strict ordering given by μ_i, it follows that

$$(nk - p)[\chi_{X_i}] < np\,[f] < (nk + p)[\chi_{X_i}]. \tag{15}$$

Applying the state ω gives

$$nk - p < np\,\omega([f]) < nk + p. \tag{16}$$

Dividing by np gives

$$|\omega([f]) - (k/p)| \leq \frac{1}{n}. \tag{17}$$

Thus

$$|\omega([f]) - \mu_i(f)| \leq 2/n < \epsilon. \tag{18}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\omega([f]) = \mu_i(f)$, and thus ω is the state induced by μ_i, proving that G_i has a unique state.

Since $\sigma(X_i) = X_{i+1 \bmod N}$, then

$$\operatorname{supp} f \subset X_i \implies \operatorname{supp}(\mathcal{L}f) \subset \sigma(\operatorname{supp}[f]) \subset X_{i+1 \bmod N}. \tag{19}$$

It follows that $\mathcal{L}_* G_i \subset G_{i+1 \bmod N}$. Since σ is surjective, then $\mathcal{L}_*$ is surjective on G ([32, Lemma 3.5]), so we must have $\mathcal{L}_* G_i = G_{i+1 \bmod N}$ for each i.

Finally, suppose that μ is scaled by the factor s, and let ω be any state on $DG(\tau)$ that is scaled by a factor s_1. Fix an index i. By the uniqueness of states on G_i, there is a constant k such that $\omega = k\mu$ on G_i. Then $\omega \circ \mathcal{L}_*^N = k\mu \circ \mathcal{L}_*^N$ on G_i, so

$s_1^N \omega = ks^N \mu$ on G_i. If $\omega|G_i$ were 0, then the fact that ω is scaled by $\mathcal{L}_*$ and that $G_1, \ldots, G_N$ are cyclically permuted by $\mathcal{L}_*$ would imply that $\omega = 0$. Therefore we conclude that $s_1 = s$. Now it follows that $\omega \circ \mathcal{L}_*^j = k\mu \circ \mathcal{L}_*^j$ on G_i for $j = 1, \ldots, N$, so $\omega = k\mu$ on all G_j. Then $1 = \omega(1) = k\mu(1)$ implies that $k = 1$, so $\omega = \mu$.

The fact that the scaling factor is $s = \exp h_\tau$ follows from Corollary 3.8. □

Theorem 5.3 provides a description of all states on $DG(\tau)$. For each i, define a state $\widetilde{\mu}_i$ on $DG(\tau)$ by $\widetilde{\mu}_i([f]) = \mu(X_i)^{-1}\mu(f\chi_{X_i})$. Then $\widetilde{\mu}_i|G_i = \mu_i$, and every state on $DG(\tau)$ is a unique convex combination of $\widetilde{\mu}_1, \ldots, \widetilde{\mu}_N$, so the state space is a simplex with $\widetilde{\mu}_1, \ldots, \widetilde{\mu}_N$ as extreme points.

Corollary 5.4. *If maps $\tau_i : I \to I$ are piecewise monotonic and transitive for $i = 1, 2$, and if the dimension triples for τ_1 and τ_2 are isomorphic, then τ_1 and τ_2 have the same topological entropy.*

Proof. Assume that the dimension triples for τ_1 and τ_2 are isomorphic. Suppose first that neither map is essentially injective. Then each dimension group has a unique scaled state (Theorem 5.3). By the isomorphism of the dimension triples, the scaling factors must coincide. Since the scaling factors are given by the exponential of the topological entropy, the two entropies must coincide.

Suppose now that τ_1 is essentially injective, and that $\sigma_1 : X^1 \to X^1$ is the associated local homeomorphism. By Proposition 4.3, τ_1 admits a scaling measure, and all such scaling measures have scaling factor $s = \exp h_{\tau_1} = 1$, so $h_{\tau_1} = 0$. By Proposition 5.2, all scaled states on $DG(\tau_1)$ have scaling factors 1. The same must hold for $DG(\tau_2)$. By Proposition 4.3 and Proposition 5.2, this implies that τ_2 is essentially injective, and so has zero topological entropy. In particular, $h_{\tau_1} = h_{\tau_2}$. □

Corollary 5.5. *If $\tau : I \to I$ is piecewise monotonic and topologically exact, then $DG(\tau)$ is simple and has a unique state, given by the unique scaling measure, and has the strict order given by that state.*

Proof. If τ is topologically exact, then so is the associated local homeomorphism $\sigma : X \to X$ (Proposition 2.7). It follows that σ is not injective, and so τ is not essentially injective (Lemma 4.2). Now the conclusion follows from Theorem 5.3. □

Renault [27, Example 6.2] has proven a similar uniqueness result for states on dimension groups associated with positively expansive maps that are topologically exact. (See the remarks on positively expansive maps following Theorem 4.5.)

Definition 5.6. Let G be a dimension group with distinguished order unit e. An element $x \in G$ is *infinitesimal* if for all $n \geq 0$, $-e \leq n[f] \leq e$. (This is independent of the choice of order unit.) The subgroup of infinitesimals is denoted G_{inf}.

Observe that if G is a dimension group with distinguished order unit e, and $\Phi : G \to G$ is a positive endomorphism, then G_{inf} will be invariant under Φ.

If G is an ordered abelian group, and H is a subgroup that is an order ideal (i.e., $g \in G$ and $0 \leq g \leq h \in H$ imply $g \in H$), then we define $(G/H)^+$ to be the image under the quotient map of the positive cone of G. With the ordering given by $(G/H)^+$, G/H becomes an ordered abelian group.

Below we view $\mathbb{R}^N$ as an ordered group with the usual coordinatewise order; it is evidently a dimension group.

Proposition 5.7. *Let $\tau : I \to I$ be piecewise monotonic and transitive and not essentially injective, let $\sigma : X \to X$ be the associated local homeomorphism, and let $DG(\tau) = \oplus_i G_i$ be the decomposition given in Theorem 5.3. Let μ be the unique measure on X scaled by σ, with scaling factor s. Then we have an isomorphism of dimension groups given by*

$$DG(\tau)/DG(\tau)_{\mathrm{inf}} \cong \oplus_i G_i/(G_i)_{\mathrm{inf}} \cong \oplus_i \mu(G_i), \tag{20}$$

where each $G_i/(G_i)_{\mathrm{inf}}$ has the quotient ordering, and $\oplus_i \mu(G_i)$ has the ordering inherited from $\mathbb{R}^N$. An isomorphism from $DG(\tau)/DG(\tau)_{\mathrm{inf}}$ onto $\oplus_i \mu(G_i)$ is given by $\Psi([f]) = \oplus_i \mu(f\chi_{X_i})$, and Ψ carries the order automorphism induced by $\mathcal{L}_*$ on $DG(\tau)/DG(\tau)_{\mathrm{inf}}$ to the order automorphism of $\oplus_i \mu(G_i)$ given by

$$(\lambda_1, \lambda_2, \dots \lambda_N) \mapsto (s\lambda_N, s\lambda_1, \dots, s\lambda_{N-1}). \tag{21}$$

Proof. If $X_1, \dots, X_N$ is the partition of X given in Theorem 4.5, then $[f] \in DG(\tau)$ is an infinitesimal in $DG(\tau)$ iff each $[f\chi_{X_i}]$ is infinitesimal in G_i, so $G_{\mathrm{inf}} = \oplus_i (G_i)_{\mathrm{inf}}$. Since $G = \oplus_i G_i$, the first isomorphism in (20) follows. For the second isomorphism, since G_i has the strict ordering from $\mu|G_i$, then $[f] \mapsto \mu(f)$ is a positive homomorphism from G_i onto $\mu(G_i)$ with kernel $(G_i)_{\mathrm{inf}}$, and the quotient ordering is exactly the ordering on $\mu(G_i)$ inherited from $\mathbb{R}$.

To verify the final statement of the proposition, for $f \in C(X, \mathbb{Z})$ and $1 \le i \le N$, let $f_i = f\chi_{X_i}$, and $\lambda_i = \mu(f_i)$. Then $\Psi([f]) = (\lambda_1, \dots, \lambda_N)$. Since $\operatorname{supp} \mathcal{L} f_i \subset X_{i+1 \bmod N}$ for all i, then the kth coordinate of $[\mathcal{L} f]$ in $\oplus_i G_i$ is $[\mathcal{L} f_{k-1 \bmod N}]$, so

$$(\Psi([\mathcal{L} f]))_k = \mu(\mathcal{L} f_{k-1 \bmod N}) = s\mu(f_{k-1 \bmod N}) = s\lambda_{k-1 \bmod N}, \tag{22}$$

which completes the proof of the proposition. □

Krieger proved a result analogous to Proposition 5.7 with two-sided irreducible shifts of finite type in place of transitive piecewise monotonic maps τ, and with the invariant measure of maximal entropy in place of the scaling measure μ ([20, Thm. 3.2]). He then used the range of this measure on the clopen subsets as an invariant to distinguish certain shifts of finite type. The desire to generalize this result to interval maps was one motivation for the current paper.

6. Dimension triples as invariants for piecewise monotonic maps

The dimension triple determines the topological entropy (Corollary 5.4). However, to determine τ up to conjugacy, the dimension triple is still not quite enough, and we will now discuss some additional information that suffices.

Definition 6.1. Let $\tau : I \to I$ be piecewise monotonic, continuous, and transitive, with associated partition $0 = a_0 < a_1 < \dots < a_n = 1$. The n-tuple $(I(a_0, a_1), I(a_1, a_2), \dots, I(a_{n-1}, a_n))$ is called the *canonical sequence of generators for the dimension module* $DG(\tau)$. We will call (a_0, a_1) the *first interval of monotonicity.*

Theorem 6.2. *Let $\tau_1 : I \to I$ and $\tau_2 : I \to I$ be continuous, transitive, piecewise monotonic maps. Then there is an increasing conjugacy from τ_1 to τ_2 iff*

- (i) *the maps τ_1 and τ_2 are both increasing, or both decreasing, on their first interval of monotonicity, and*
- (ii) *there is a unital isomorphism Φ from the dimension triple for τ_1 onto the dimension triple for τ_2, taking the canonical sequence of generators for $DG(\tau_1)$ onto that for $DG(\tau_2)$.*

Proof. Assume that (i) and (ii) hold. For $i = 1, 2$, let $\sigma_i : X^i \to X^i$ be the local homeomorphism associated with τ_i. Note that since each $\tau_i : I \to I$ is continuous and transitive, it is surjective, and is not essentially injective. (If it were essentially injective, by continuity it would be injective, and thus would be a homeomorphism. However, no homeomorphism of I is transitive.) Thus by Corollary 4.6 and Proposition 3.3, for $i = 1, 2$, there is a unique measure μ_i on $[0, 1]$ scaled by τ_i by a factor s_i, with associated scaled measure $\widetilde{\mu}_i$ on X^i. Define $h_i : [0, 1] \to [0, 1]$ by $h_i(x) = \mu_i([0, x])$. Then h_i is an increasing conjugacy from τ_i onto the uniformly piecewise linear map $T_i = h_i \circ \tau_i \circ h_i^{-1}$, with slopes $\pm s_i$ for $i = 1, 2$. (See the proof of Proposition 3.6). We will show that $T_1 = T_2$, and then $h_2^{-1} \circ h_1$ will be the desired increasing conjugacy from τ_1 to τ_2.

If $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are the partitions associated with τ_1 and τ_2 respectively, then the lengths of the i-th interval of monotonicity for T_1 and T_2 will be $\mu_1(a_{i-1}, a_i) = \widetilde{\mu}_1(I(a_{i-1}, a_i))$ and $\mu_2(I(b_{i-1}, b_i)) = \widetilde{\mu}_2(I(b_{i-1}, b_i))$ respectively. By the uniqueness of the scaled states on $DG(\tau_1)$ and $DG(\tau_2)$, Φ must take $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$, and we must have $s_1 = s_2$. Since Φ takes the canonical sequence of generators of $DG(\tau_1)$ to the corresponding sequence for $DG(\tau_2)$, then $\widetilde{\mu}_1(I(a_{i-1}, a_i)) = \widetilde{\mu}_2(I(b_{i-1}, b_i))$ for all i. Thus T_1 and T_2 have intervals of monotonicity of the same length, increase and decrease on these intervals in the same order, and have the same slopes. Thus for all x, we have $T_2(x) = T_1(x) + T_2(0) - T_1(0)$. Since $\max_{x \in I} T_1(X) = \max_{x \in I} T_2(x) = 1$, we must have $T_1(0) = T_2(0)$. Thus $T_1 = T_2$, which completes the proof that τ_1 and τ_2 are conjugate.

Conversely, if there is an increasing conjugacy h from τ_1 to τ_2, then (i) will hold, and the conjugacy will carry each interval (a_{i-1}, a_i) to the corresponding interval (b_{i-1}, b_i). From the construction of the associated local homeomorphisms $\sigma_i : X^i \to X^i$, h will lift to a conjugacy $\widetilde{h}$ from (X^1, σ_1) onto (X^2, σ_2), carrying $I(a_{i-1}, a_i)$ to $I(b_{i-1}, b_i)$. Then $[f] \mapsto [f \circ \widetilde{h}^{-1}]$ will be the desired isomorphism of dimension group triples. □

We will see in Example 9.3 that condition (i) in Theorem 6.2 is not redundant.

Define $\phi : I \to I$ by $\phi(x) = 1 - x$. If $\tau_1 : I \to I$ is any piecewise monotonic map, and $\tau_2 = \phi \circ \tau_1 \circ \phi^{-1}$, then ϕ will induce an isomorphism from $DG(\tau_1)$ onto $DG(\tau_2)$ that will carry the sequence of canonical generators of $DG(\tau_1)$ onto the sequence of canonical generators of $DG(\tau_2)$ in reverse order. Since every decreasing homeomorphism is the composition of an increasing homeomorphism with ϕ, it isn't difficult to reformulate Theorem 6.2 to include arbitrary (increasing or decreasing) conjugacies; we leave that to the reader.

For particular families of interval maps, it can happen that the dimension triple determines the map among members of that family. For example, we will see later that transitive unimodal maps are determined up to conjugacy by their entropy, which, as we have seen, can be recovered from the dimension triple.

For a related discussion of dimension groups as an invariant for minimal homeomorphisms of a Cantor set, see [12], where it is shown, for example, that two such homeomorphisms are orbit equivalent iff certain associated dimension groups modulo infinitesimals are unitally order isomorphic.

In the remainder of this paper, we will use the results of previous sections to compute the dimension groups for several families of transitive maps. For later reference, some of these are illustrated in Figure 1.

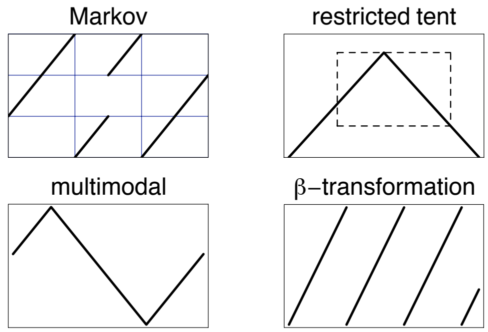

FIGURE 1. Examples of transitive piecewise monotonic maps

7. MARKOV MAPS

For use in later sections, we review the description of $DG(\tau)$ for Markov maps τ from section 8 of [32]. A piecewise monotonic map $\tau : I \to I$ is Markov if there is a partition $B = \{0 = b_0, b_1, \ldots, b_n = 1\}$ of I such that τ is monotonic on each interval $[b_{i-1}, b_i]$, and such that the image of each such interval is a union of intervals of the form $[b_{j-1}, b_j]$. We also require that the orbits under $\widehat{\tau}$ of the critical points and endpoints eventually land in B. Observe that any Markov map is necessarily eventually surjective, since the successive images of the whole space decrease and each image is a union of some of the Markov partition intervals.

Definition 7.1. If A is an $n \times n$ zero-one matrix, G_A denotes the stationary inductive limit $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$ in the category of ordered abelian groups. We represent members of G_A as equivalence classes $[v, n]$ where $v \in \mathbb{Z}^n$ and $n \in N$, with $[v_1, n_1] = [v_2, n_2]$ if $v_1 A^{n_2+k} = v_2 A^{n_1+k}$ for some $k \geq 0$. G_A will be a dimension group. The action of A induces an automorphism of G_A denoted A_*, where $A_*[v, n] = [vA, n]$, so we view G_A as a $\mathbb{Z}[t, t^{-1}]$ module, and refer to (G_A, G_A^+, A_*) as the dimension triple associated with A.

Proposition 7.2. ([32, Prop. 8.4]) *Let $\tau : I \to I$ be piecewise monotonic and Markov, with associated local homeomorphism $\sigma : X \to X$. Let $E_1, E_2, \ldots, E_q$ be the associated Markov partition for σ, with incidence matrix A, and define $\psi : \mathbb{Z}^q \to C(X, \mathbb{Z})$ by $\psi(z_1, z_2, \ldots, z_q) = \sum_i z_i E_i$. Then the map $\Phi : G_A \to DG(\tau)$ defined by $\Phi([v, n]) = \mathcal{L}_*^{-n}[\psi(v)]$ is an isomorphism from the dimension triple (G_A, G_A^+, A_*) onto the dimension triple $(DG(\tau), DG(\tau)^+, \mathcal{L}_*)$.*

Example 7.3. Let τ be the Markov map in Figure 1, and let $\sigma : X \to X$ be the associated local homeomorphism. Then τ is Markov, and the associated incidence matrix A is primitive. If (X_A, σ_A) is the associated shift of finite type, by [32, Cor. 8.9 and Ex. 8.10], (X, σ) is conjugate to (X_A, σ_A), and the dimension triples $(DG(\tau), DG(\tau)^+, \mathcal{L}_*)$ and (G_A, G_A^+, A_*) are isomorphic. The unique measure scaled by τ is Lebesgue measure m, with scaling factor 2. If $E_1 = I(0, 1/3)$, $E_2 = I(1/3, 2/3)$, $E_3 = I(2/3, 1)$ are the three intervals of the Markov partition, then the image of the unique state is $\sum_i \mathbb{Z}[s, s^{-1}]m(E_i) = (1/3)\mathbb{Z}[1/2]$. By Proposition 5.7, we also have $DG(\tau)/DG(\tau)_{\text{inf}} \cong (1/3)\mathbb{Z}[1/2]$.

When τ is piecewise monotonic, topologically exact, and Markov, with incidence matrix A, it is interesting to know whether the unique state gives an isomorphism of $DG(\tau) \cong G_A$ onto the range of the state, i.e., when there are no infinitesimals. By [6, Thm. 5.10 and Cor. 5.11], this happens iff the characteristic polynomial $p(t)$ of the primitive matrix A, with the largest power of t divided out, is irreducible. In Example 7.3, the characteristic polynomial of the incidence matrix A is $p(t) = t^3 - 3t - 2 = (t+1)(t^2 - t - 2)$. Since p is not irreducible, then there are infinitesimals.

8. Unimodal maps

A *unimodal map* $\tau : I \to I$ is a continuous piecewise monotonic map with just two intervals of monotonicity $[0, c]$ and $[c, 1]$, such that τ increases on $[0, c]$ and decreases on $[c, 1]$. In this section we will describe all transitive unimodal maps, and compute their dimension triples.

Given $s > 1$, we define the *restricted tent map* T_s by

$$T_s(x) = \begin{cases} 1 + s(x - c) & \text{if } x \le c \\ 1 - s(x - c) & \text{if } x > c, \end{cases} \tag{23}$$

where $c = 1 - 1/s$. This is the usual symmetric tent map τ_0 on $[0, 1]$ with slopes $\pm s$, restricted to the interval $[\tau_0^2(1/2), \tau_0(1/2)]$, which is the interval of most interest for the dynamics. Then the map has been rescaled so that its domain is [0,1] (See Figure 1.) Note that $T_s(c) = 1$ and $T_s(1) = 0$. The following is essentially in [16] and [22, p. 245].

Lemma 8.1. *Let $T = T_s$ be the restricted tent map, with slopes $\pm s$.*

(i) *If $\sqrt{2} < s \le 2$, then T is topologically mixing (and exact).*

(ii) *If $s = \sqrt{2}$, and p is the fixed point of T, then T exchanges $[0, p]$ and $[p, 1]$, and T^2 is conjugate to the full tent map on each of these intervals. Thus T is transitive, but is not topologically exact.*

(iii) *If $s < \sqrt{2}$, then T is not transitive.*

Proof. Let $c = 1 - 1/s$, so that c is the critical point of $T = T_s$, and let p be the fixed point of T. If $s < \sqrt{2}$, then $0 < c < T^2(0) < p < T(0) < T^3(0) < 1$, so that $[0, T^2(0)] \cup [T(0), 1]$ is a closed invariant subset, with non-empty interior, contradicting transitivity. If $\sqrt{2} < s \le 2$, then for any interval J, either J and $T(J)$ contain c (in which case $T^2(J)$ contains 0 and 1, so $T^2(J) = [0, 1]$), or else applying T^2 to J gives an interval whose length is at least $s(s/2) = s^2/2 > 1$ times the length of J. Thus for some n, $T^n(J) = [0, 1]$. If $s = \sqrt{2}$, then $T(0)$ is fixed by T, and the statements in the proposition for $s = \sqrt{2}$ are straightforward to check. □

Recall that $\mathbb{Z}[t, t^{-1}]$ denotes the additive group of Laurent polynomials with integer coefficients. The group $\mathbb{Z}[t, t^{-1}]$ has no given order. For each positive $s \in \mathbb{R}$, we write $\widehat{s} : \mathbb{Z}[t, t^{-1}] \to \mathbb{R}$ for the evaluation map, i.e., $\widehat{s}(p) = p(s)$. For each such s, we define a (partial) order on $\mathbb{Z}[t, t^{-1}]$ by $p \ge 0$ iff $p = 0$ or $p(s) > 0$, and call this order the *strict order given by evaluation at s*.

For $0 < s \in \mathbb{R}$, we denote by $\mathbb{Z}[s, s^{-1}]$ the subgroup of $\mathbb{R}$ generated by $\{s^n \mid n \in \mathbb{Z}\}$, or equivalently, the subring of $\mathbb{R}$ generated by s and s^{-1}. We equip $\mathbb{Z}[s, s^{-1}]$ with the order inherited from $\mathbb{R}$.

We summarize results from [32, Thm. 9.1] for $DG(\tau)$ when τ is unimodal and surjective. Analysis of dimension groups for unimodal maps divides naturally into the cases where the map is Markov or not. If the critical point c is eventually periodic, then τ will be Markov, so $(DG(\tau), DG(\tau)^+, \mathcal{L}_*)$ will be isomorphic to (G_A, G_A^+, A_*). If instead the orbit of c is infinite, then $p \mapsto p(\mathcal{L}_*)I(0,1)$ is an isomorphism of $\mathbb{Z}[t,t^{-1}]$ onto $DG(\tau)$ as an abelian group, with the action of $\mathcal{L}_*$ on $DG(\tau)$ corresponding to multiplication by t on $\mathbb{Z}[t,t^{-1}]$. (Here I(0,1) is the class in $DG(\tau)$ of the function identically 1 on X; see the remark before Theorem 2.12.)

If τ is transitive, we now describe the order on $DG(\tau)$ in the case where c is not eventually periodic.

Proposition 8.2. *Let $\tau : I \to I$ be unimodal and transitive, and let $s = \exp h_\tau$. Then $\sqrt{2} \le s \le 2$, and τ is conjugate to the restricted tent map T_s. If in addition the critical point c is not eventually periodic, then*

- (i) *The map $p \mapsto p(\mathcal{L}_*)I(0,1)$ is a group isomorphism of $\mathbb{Z}[t,t^{-1}]$ onto $DG(\tau)$. This isomorphism carries the evaluation map $\widehat{s} : \mathbb{Z}[t,t^{-1}] \to \mathbb{R}$ to the unique scaled state on $DG(\tau)$.*
- (ii) *If $s > \sqrt{2}$, and we give $\mathbb{Z}[t,t^{-1}]$ the strict order given by evaluation at s, then the group isomorphism in (i) is also an order isomorphism.*
- (iii) *If $s > \sqrt{2}$ then the unique state on $DG(\tau)$ induces a group and order isomorphism from $DG(\tau)/DG(\tau)_{\inf}$ onto $\mathbb{Z}[s,s^{-1}] \subset \mathbb{R}$, with the automorphism of $DG(\tau)/DG(\tau)_{\inf}$ induced by $\mathcal{L}_*$ carried to multiplication by s.*

Proof. By Corollary 4.4, τ is conjugate to a piecewise linear map T with slopes $\pm s$, with $s = \exp h_\tau$. If $T(1) > 0$, then $T(0) = 0$, and for $\epsilon > 0$ small enough, and $J = (\epsilon/2, \epsilon)$, $T^n(J) \cap J = \emptyset$ for all $n \ge 1$. This would imply that τ is not transitive, so we must have $T(1) = 0$. Then $T = T_s$, and $\sqrt{2} \le s \le 2$ by Lemma 8.1.

Assume now that the orbit of c is not eventually periodic. As discussed in the remarks preceding this proposition, the map $p \to p(\mathcal{L}_*)I(0,1)$ is an isomorphism of $\mathbb{Z}[t,t^{-1}]$ onto $DG(\tau)$. Since τ is transitive, there is a unique scaled state ω on $DG(\tau)$ (Theorem 5.3). By the scaling property, $\omega(p(\mathcal{L}_*)I(0,1)) = p(s) = \widehat{s}(p)$. Thus the isomorphism $p \mapsto p(\mathcal{L}_*)I(0,1)$ carries $\widehat{s}$ to ω.

When $\sqrt{2} < s \le 2$, then T_s is topologically exact (Lemma 8.1), and since τ is conjugate to T_s, then τ is topologically exact. Hence by Corollary 5.5 there is just one state on $DG(\tau)$, and $DG(\tau)$ is a simple dimension group with the strict order from this state. Therefore by (i), if we give $\mathbb{Z}[t,t^{-1}]$ the strict order given by evaluation at s, then $p \mapsto p(\mathcal{L}_*)I(0,1)$ becomes an order isomorphism as well as a group isomorphism.

By exactness of τ and Proposition 5.7, the unique state ω on $DG(\tau)$ is a group and order isomorphism of $DG(\tau)/DG(\tau)_{\inf}$ onto $\omega(DG(\tau)) \subset \mathbb{R}$. Since this unique state must coincide with the unique scaled state, which has scaling factor s, for $p \in \mathbb{Z}[t,t^{-1}]$ we have $\omega(p(\mathcal{L}_*)I(0,1)) = p(s)$, so $\omega(DG(\tau)) \supset \mathbb{Z}[s,s^{-1}]$. Since $p \to p(\mathcal{L}_*)I(0,1)$ maps $\mathbb{Z}[t,t^{-1}]$ onto $DG(\tau)$, the image of ω equals $\mathbb{Z}[s,s^{-1}]$. The fact that ω carries the action of $\mathcal{L}_*$ to multiplication by s is just a restatement of the fact that ω is scaled by $\mathcal{L}_*$ by the factor s. □

In Proposition 8.2, if $s = \sqrt{2}$, the dimension group for T_s is the direct sum of two copies of the dimension group for T_2. (See Lemma 8.1 and Theorem 5.3). The dimension triples for T_2 and $T_{\sqrt{2}}$ are computed in [32, Examples 9.2, 9.3].

Corollary 8.3. *Two unimodal transitive maps are conjugate iff their dimension triples are isomorphic.*

Proof. By Corollary 5.4, two transitive piecewise monotonic maps with isomorphic dimension triples have the same topological entropy. By Proposition 8.2, transitive unimodal maps with the same topological entropy are conjugate. □

Example 8.4. Let τ be the restricted tent map T_s, with $\sqrt{2} < s < 2$. Since τ is topologically exact, then $DG(\tau)$ is simple (Corollary 5.5). If s is transcendental, then $p(s)$ is non-zero for all $p \in \mathbb{Z}[t, t^{-1}]$, so the unique state is an order automorphism of $DG(\tau) = \mathbb{Z}[t, t^{-1}]I(0, 1)$ onto $\mathbb{Z}[s, s^{-1}] \subset \mathbb{R}$.

Example 8.5. Let $s = 3/2$. Then the critical point of the restricted tent map T_s is not eventually periodic, cf. [32, Example 9.4]. By Proposition 8.2, $DG(T_s) \cong \mathbb{Z}[t, t^{-1}]$, with the strict order given by the unique state, namely evaluation at s. The image of this state is $\mathbb{Z}[s, s^{-1}]$, but this state is not 1-1, e.g. $p(t) = 2t - 3$ is sent to zero. Thus there are infinitesimals in this dimension group.

9. Multimodal maps

In [32, Prop. 10.6], sufficient conditions are given for a continuous n-modal map to have a dimension group isomorphic as an (unordered) module to $(\mathbb{Z}([t, t^{-1}])^{n-1}$. We will see that if we assume the maps are mixing and uniformly piecewise linear, we can give an explicit description of the (ordered) dimension group modulo the subgroup of infinitesimals. With some extra assumptions that guarantee that there are no infinitesimals, we can describe explicitly the dimension module itself. Since every continuous mixing piecewise monotonic map is conjugate to a uniformly piecewise linear map, these results are also applicable to such maps.

Recall that a continuous topologically mixing piecewise monotonic map is topologically exact, cf. [26, Thm. 2.5] or Proposition 4.9.

Proposition 9.1. *Let $\tau : I \to I$ be continuous, piecewise linear, and topologically mixing, with slopes $\pm s$, and with associated partition $a_0 < a_1 < \cdots < a_n$. Then $DG(\tau)/DG(\tau)_{\text{inf}} \cong \sum_{i=1}^{n} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$ (with the order inherited from $\mathbb{R}$, and with the action of $\mathcal{L}_*$ given by multiplication by s).*

Proof. By Corollary 5.5, there is a unique state, which is given by the unique scaling measure, namely, Lebesgue measure. By Corollary 2.13, the intervals $I(a_{i-1}, a_i)$ for $1 \le i \le n$ generate the dimension module $DG(\tau)$. By Proposition 5.7, the ordered group $DG(\tau)/DG(\tau)_{\text{inf}}$ is isomorphic to the range of that state, namely, $\sum_i \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$. □

Proposition 9.2. *Let $\tau : I \to I$ be continuous, piecewise linear, and topologically mixing, with slopes $\pm s$, with associated partition $a_0 < a_1 < \cdots < a_n$. Assume*

- (i) *s is transcendental,*
- (ii) *the lengths of the intervals $[a_0, a_1], \ldots, [a_{n-2}, a_{n-1}]$ are independent over $\mathbb{Z}[s, s^{-1}]$,*
- (iii) *$\tau(\{0, 1\}) \cap \{0, 1\} = \emptyset$.*

Then the unique state on $DG(\tau)$ is an order isomorphism from $DG(\tau)$ onto the subgroup $\sum_{i=1}^{n-1} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$ of $\mathbb{R}$, where the subgroup is given the order inherited from $\mathbb{R}$. This isomorphism carries the action of $\mathcal{L}_*$ to multiplication by s.

Proof. Since τ is uniformly piecewise linear with slopes $\pm s$, then Lebesgue measure is scaled by τ by the factor s, and thus induces a state ω on $DG(\tau)$ which is scaled by $\mathcal{L}_*$ by the factor s (Proposition 3.3 and Proposition 5.2). As observed before

Proposition 9.1, τ will be topologically exact. Therefore there is a unique state ω on $DG(\tau)$, which must coincide with the state given by Lebesgue measure, and the order on $DG(\tau)$ is the strict order from that state, cf. Corollary 5.5.

We are going to show that ω is a group and order isomorphism from $DG(\tau)$ onto the subgroup $\sum_{i=1}^{n-1} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$ of $\mathbb{R}$. We first show ω maps $DG(\tau)$ onto this subgroup. We begin by showing that $DG(\tau)$ is generated as a $\mathbb{Z}[t, t^{-1}]$ module by $E_1, E_2, \ldots, E_{n-1}$. Let $E_i = I(a_{i-1}, a_i)$ for $1 \leq i \leq n$. By Corollary 2.13, $E_1, E_2, \ldots, E_n$ generate the dimension module. (Recall that we identify intervals with the equivalence classes in $DG(\tau)$ of their characteristic functions.) By the assumption (iii), if $\tau(a_x) = 0$ and $\tau(a_y) = 1$ with $x < y$, then $\sum_{i=x}^{y-1} \pm\mathcal{L}_*(I(a_i, a_{i+1})) = I(0, 1)$, with appropriate choices of the $\pm$ signs. (See [32, proof of Prop. 10.6] for details.) Thus $I(0, 1)$ is in the submodule generated by $E_1, \ldots, E_{n-1}$. Since $E_n = I(0, 1) - \sum_{i=1}^{n-1} E_i$, then E_n also is in the submodule generated by $E_1, E_2, \ldots, E_{n-1}$, so the latter elements generate $DG(\tau)$.

Thus every element of $DG(\tau)$ has the form $\sum_{i=1}^{n-1} p_i(\mathcal{L}_*)E_i$, where $p_1, \ldots, p_{n-1} \in \mathbb{Z}[t, t^{-1}]$. Since ω is scaled by $\mathcal{L}_*$ by the factor s, we have

$$\omega(\sum_{i=1}^{n-1} p_i(\mathcal{L}_*)E_i) = \sum_{i=1}^{n-1} p_i(s)\omega(E_i) = \sum_{i=1}^{n-1} p_i(s)(a_i - a_{i-1}) \in \sum_{i=1}^{n-1} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1}).$$

It follows that ω maps $DG(\tau)$ onto $\sum_{i=1}^{n-1} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$.

Next we show the kernel of ω is zero. Since s is transcendental, if $0 \neq p \in \mathbb{Z}[t, t^{-1}]$, then $p(s)$ is non-zero. If $p_1, \ldots, p_{n-1} \in \mathbb{Z}[t, t^{-1}]$ and $g = \sum_{i=1}^{n-1} p_i(\mathcal{L}_*)E_i$, then $\omega(g) = \sum_{i=1}^{n-1} p_i(s)(a_i - a_{i-1})$, which by (ii) is non-zero unless all $p_i(s)$ are zero. Since s is transcendental, this can happen only if all p_i are zero. Thus the kernel of ω is zero.

We've shown ω is a group isomorphism from $DG(\tau)$ onto $\sum_{i=1}^{n-1} \mathbb{Z}[s, s^{-1}](a_i - a_{i-1})$. Since $DG(\tau)$ has the strict order from ω, then ω is also an order isomorphism. Finally, since ω is scaled by $\mathcal{L}_*$ by the factor s, by the definition of scaling (see above Proposition 5.2), ω carries the action of $\mathcal{L}_*$ to multiplication by s, which completes the proof. □

If $\tau : I \to I$ is continuous, mixing, and piecewise monotonic, then τ is conjugate to a uniformly piecewise linear map whose slopes are $\pm s$ with $s = \exp h_\tau$, cf. Corollary 4.4. If $a_0, a_1, \ldots, a_n$ is the partition associated with τ, and μ is the unique measure on I scaled by τ, then the conjugacy can be chosen to carry (a_{i-1}, a_i) onto an interval of length $\mu([a_{i-1}, a_i])$, cf. Proposition 3.6. Thus Propositions 9.1 and 9.2 are valid for such a map τ if s is taken to be $\exp(h_\tau)$, and $(a_i - a_{i-1})$ is replaced by $\mu([a_{i-1}, a_i])$.

Example 9.3. Let $\tau : I \to I$ be the uniformly piecewise linear map in Figure 1 on page 19, with slopes $\pm s$, with s transcendental, and $2 < s < 3$. Since $s > 2$, under iteration, the length of any interval expands until the interval contains both critical points, and then the next iterate equals I. Thus τ is topologically exact. Assume that the first critical point is at $\alpha \notin \mathbb{Z}[s, s^{-1}]$. Then by Proposition 9.2, the dimension module is order isomorphic to $\alpha\mathbb{Z}[s, s^{-1}] + \mathbb{Z}[s, s^{-1}]$. Now let τ' be the piecewise monotonic map whose graph is that of τ turned upside down, i.e., $\tau'(x) = 1 - \tau(x)$. Then the same argument shows that τ' is topologically exact, and that the dimension module for τ' also is order isomorphic to $\alpha\mathbb{Z}[s, s^{-1}] + \mathbb{Z}[s, s^{-1}]$.

Yet τ and τ' are not conjugate. (For example, they have different numbers of fixed points.) Thus condition (i) in Theorem 6.2 is not redundant.

10. β-TRANSFORMATIONS

If $\beta > 1$, the *β-transformation* on $[0, 1)$ is the map $\tau_\beta : x \mapsto \beta x \bmod 1$. (See Figure 1 on page 19.) We extend τ_β to $[0, 1]$ by defining $\tau_\beta(1) = \lim_{x\to 1^-} \tau_\beta(x)$. Then τ_β is piecewise monotonic, and the associated partition is $\{0, 1/\beta, 2/\beta, \ldots, n/\beta, 1\}$, where $n = [\beta]$ is the greatest integer $\leq \beta$.

Katayama, Matsumoto, and Watatani have associated C*-algebras F^∞_β and O_β with the β-shift. The algebra F^∞_β is a simple AF-algebra ([17, Prop. 3.5]), so $K_0(F^\infty_\beta)$ is a simple dimension group. In this section, we will compute the dimension group for τ_β, and will show that the dimension groups $DG(\tau_\beta)$ and $K_0(F^\infty_\beta)$ are isomorphic as abelian groups, and are also order isomorphic if 1 is eventually periodic.

Lemma 10.1. *For $\beta > 1$, the β-transformation τ is topologically exact, and thus $DG(\tau)$ is a simple dimension group. There is a unique state, given by Lebesgue measure, and scaled by $\mathcal{L}_*$ by the factor β. The order on $DG(\tau)$ is the strict order given by this state.*

Proof. If J is any interval whose interior does not contain one of the partition points k/β for $1 \leq k \leq n$, (where n is the greatest integer $\leq \beta$), then applying τ multiplies the length of J by β. Thus for n large enough, the interior of $\tau^n(J)$ includes one of these partition points, and then the interior of $\widehat{\tau}^{n+1}(J)$ includes zero. Succeeding images will contain 0 as an interior point, and the component of the images containing 0 will grow in length until it includes $[0, 1]$. Thus τ is exact. Simplicity of $DG(\tau)$ follows from [32, Cor. 5.4]. Since Lebesgue measure is scaled by τ by the factor β, then it induces a state, scaled by $\mathcal{L}_*$ by the factor β, cf. Proposition 5.2. That there is a unique state, and that $DG(\tau)$ has the strict order from this state, follow from exactness of τ and Corollary 5.5. $\square$

If $x \in I$, and $n = [\beta]$, we give the intervals $[0, 1/\beta)$, $[1/\beta, 2/\beta)$, ..., $[(n-1)/\beta, n/\beta)$, $[n/\beta, 1]$ the labels $0, 1, \ldots, n$. (If $\beta = n$, there are $n + 1$ intervals, with the last one being the singleton $\{1\}$.) We define the *itinerary* of $x \in I$ to be the sequence $n_0n_1n_2\ldots$, where $\tau^k x$ is in the interval with label n_k. (Warning: in general, this is not a Markov partition, so this is not an itinerary map in the sense used in [32].)

If τ_β is Markov (cf. §7), then by definition, the orbit of each endpoint of each interval of monotonicity under $\hat{\tau}_\beta$ must be finite, and so in particular 1 has a finite orbit, and hence is eventually periodic. Conversely, if 1 is eventually periodic, then the orbit of 1 together with the points 0, $1/\beta$, ..., n/β give a Markov partition. Thus the map τ_β will be Markov iff 1 is eventually periodic. In that case, if A is the associated incidence matrix, the dimension triple for τ_β is isomorphic to (G_A, G_A^+, A_*), cf. Definition 7.1 and Proposition 7.2. In Proposition 10.3, we will express G_A as an inductive limit with respect to right multiplication on $\mathbb{Z}^q$ by a matrix $B_\mathcal{L}$ whose entries are expressed in terms of the itinerary of 1, and will describe the unique state.

Lemma 10.2. *Let τ be the β-transformation, and $\sigma : X \to X$ the associated local homeomorphism. Assume 1 is eventually periodic, with $1, \tau 1, \ldots, \tau^{p-1}1$ distinct,* *and with $\tau^p 1 = \tau^k 1$ for some $k < p$. Let the itinerary of 1 be $n_0 n_1 n_2 \dots n_k \dots n_p \dots$, and let $M = \mathbb{Z}[t]I(0,1)$. Let $m(t)$ be the minimal polynomial for $\mathcal{L}$ restricted to M.*

(i) *If $\tau 1 = 1$, then $m(t) = t - n_0$, and $M \cong \mathbb{Z}$.*

(ii) *If $\tau 1 \neq 1$ and $\tau^p 1 \neq 0$, then $M \cong \mathbb{Z}^p$, and*

$$m(t) = t^p - n_0 t^{p-1} - n_1 t^{p-2} - \dots - n_{p-1} - (t^k - n_0 t^{k-1} - \dots - n_{k-1}).$$

(iii) *If $\tau^p 1 = 0$, then $M \cong \mathbb{Z}^{p-1}$, and*

$$m(t) = t^{p-1} - n_0 t^{p-2} - n_1 t^{p-3} - \dots - n_{p-2}.$$

Proof. If $\tau 1 = 1$, then β is an integer n_0, and $\mathcal{L}I(0,1) = n_0 I(0,1)$, so $m(t) = t - n_0$, and $M \cong \mathbb{Z}$ follow. Hereafter we assume $\tau 1 \neq 1$.

Suppose that $\tau^p 1 \neq 0$, so that $\tau^{p-1} 1 \neq 0$. The points 0, 1, $\tau 1$, $\tau^2 1$, …, $\tau^{p-1} 1$ will be distinct. Let $b_0 < b_1 < \dots < b_p$ be these points arranged in increasing order, and define $E_i = I(b_{i-1}, b_i)$ for $1 \leq i \leq p$. Then every element of M can be written uniquely as an integral combination of $E_1, \dots, E_p$, and $\sum_i z_i E_i \geq 0$ iff $(z_1, \dots, z_p) \geq 0$, so $M \cong \mathbb{Z}^p$. If instead $\tau^p 1 = 0$, then $\tau^{p-1} 1 = 0$. Then there are p distinct points among 0, 1, $\tau 1$, $\tau^2 1$, …, $\tau^{p-2} 1$, so a similar argument shows $M \cong \mathbb{Z}^{p-1}$.

Whether or not $\tau^p 1 = 0$, σ maps each interval of monotonicity onto $I(0,1)$ (except the rightmost interval), so for $1 \leq j \leq p$,

$$\mathcal{L}I(0, \tau^{j-1} 1) = n_{j-1} I(0,1) + I(0, \tau^j 1). \tag{24}$$

Thus

$$I(0, \tau^j 1) = \mathcal{L}I(0, \tau^{j-1} 1) - n_{j-1} I(0,1). \tag{25}$$

Hence, by induction,

$$I(0, \tau^j(1)) = (\mathcal{L}^j - n_0 \mathcal{L}^{j-1} - n_1 \mathcal{L}^{j-2} \cdots - n_{j-1}) I(0,1). \tag{26}$$

(ii) Now we will establish the formula in (ii) for $m(t)$. Suppose that $\tau^p 1 \neq 0$. Since $\tau^k(1) = \tau^p(1)$, from the versions of (26) with $j = k$ and $j = p$, it follows that $\mathcal{L}$ satisfies $m(\mathcal{L})I(0,1) = 0$, where m is the polynomial in (ii). It is clear from (26) that the linear span in $C(X)$ of $I(0,1)$, $\mathcal{L}I(0,1)$, …, $\mathcal{L}^{p-1} I(0,1)$ is the same as that of $I(0,1)$, $I(0, \tau 1)$, …, $I(0, \tau^{p-1} 1)$.

Since the points 1, $\tau 1$, …, $\tau^{p-1} 1$ are distinct, and none equal 0, then $I(0,1)$, $I(0, \tau 1)$, …, $I(0, \tau^{p-1} 1)$ are linearly independent in $C(X)$, so the same must be true of the functions $I(0,1)$, $\mathcal{L}I(0,1)$, …, $\mathcal{L}^{p-1} I(0,1)$. It follows that $\mathcal{L}|_M$ cannot satisfy a polynomial of degree less than p, so m as described in (ii) is the minimal polynomial for $\mathcal{L}$.

(iii) Assume $\tau^{p-1} 1 = 0$. By (26),

$$(\mathcal{L}^{p-1} - n_0 \mathcal{L}^{p-2} - n_1 \mathcal{L}^{p-3} \cdots - n_{p-2}) I(0,1) = 0.$$

In this case, $I(0,1)$, $I(0, \tau 1)$, …, $I(0, \tau^{p-2} 1)$ are linearly independent in $C(X)$, so there is no polynomial of degree less than $p - 1$ that annihilates $\mathcal{L}|_M$. Thus the minimal polynomial for $\mathcal{L}|_M$ is as described in (iii). □

The notation used below for inductive limits was introduced after Definition 7.1.

Proposition 10.3. *Let τ be the β-transformation, and assume that 1 is eventually periodic. Let $M = \mathbb{Z}[t]I(0,1)$; cf. Lemma 10.2. Let the minimal polynomial m of $\mathcal{L}$ on M be*

$$m(\lambda) = \lambda^q - a_0 \lambda^{q-1} - a_1 \lambda^{q-2} - \dots - a_{q-1}. \tag{27}$$

Then $DG(\tau)$ is isomorphic as a group to the stationary inductive limit given by right multiplication on $\mathbb{Z}^q$ by the matrix

$$B_{\mathcal{L}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{q-1} \end{pmatrix}. \tag{28}$$

The unique state ω on $DG(\tau)$ is given by Lebesgue measure. On the inductive limit, ω satisfies

$$\omega([((z_0, \ldots, z_{q-1}), n)]) = \beta^{-n} \sum_i z_i \beta^i. \tag{29}$$

The order on $DG(\tau)$ is the strict order given by this state, and the range of this state is $\mathbb{Z}[\beta, \beta^{-1}]$.

Proof. Note that $\mathcal{L}(M) \subset M$. We will prove that

(i) for each $f \in C(X, \mathbb{Z})$, there exists $n \geq 0$ such that $\mathcal{L}^n f \in M$.
(ii) $\mathcal{L}_*$ is surjective.

It will follow that $DG(\tau)$ is isomorphic to the stationary inductive limit $M \xrightarrow{\mathcal{L}} M$, cf. [32, Lemma 8.3].

Let $B = \{0, 1, \tau 1, \tau^2 1, \ldots, \tau^{q-1} 1\}$. Note that $\tau(B) \subset B$. By (26), every element of M is an integral combination of elements $I(0, b)$ with $b \in B$, and thus is contained in the subgroup generated by $I(b_i, b_j)$ for $b_i, b_j \in B$. Let $C = \{0, 1/\beta, \ldots, n/\beta, 1\}$, where $n = [\beta]$. Note that $\widehat{\tau}(C) \subset B$. By definition of I_1, the orbit of every point in I_1 eventually lands in the orbit of C, and therefore eventually lands in B. From this (i) follows. Since a Markov map is eventually surjective, then $\mathcal{L}_*$ is surjective, cf. [32, Lemma 3.5], so (ii) holds.

By Lemma 10.2, $I(0,1)$, $\mathcal{L}I(0,1)$, …, $\mathcal{L}^{q-1}I(0,1)$ are a basis for $M \cong \mathbb{Z}^q$, and with respect to this basis, $\mathcal{L}$ has the matrix $B_{\mathcal{L}}$ given above. Define $\psi : \mathbb{Z}^q \to C(X, \mathbb{Z})$ by $\psi(z_0, z_1, \ldots, z_{q-1}) = \sum_i z_i \mathcal{L}^i I(0,1)$. Then ψ is an isomorphism from $\mathbb{Z}^q$ onto $M = \mathbb{Z}[t]I(0,1)$ (as abelian groups), and satisfies

$$\psi(vB_{\mathcal{L}}) = \mathcal{L}\psi(v) \text{ for all } v \in \mathbb{Z}^q.$$

It follows that ψ induces an isomorphism Φ of the inductive limit $(G_{B_{\mathcal{L}}}, B_{\mathcal{L}})$ and the stationary inductive limit $M \xrightarrow{\mathcal{L}} M$. Thus $(DG(\tau), \mathcal{L}_*) \cong (G_{B_{\mathcal{L}}}, B_{\mathcal{L}})$. (However, since ψ is not positive, this is not an order isomorphism.)

Recall that $DG(\tau)$ has a unique state, given by Lebesgue measure μ, and has the strict order given by that state (Lemma 10.1). If we equip $(G_{B_{\mathcal{L}}}, B_{\mathcal{L}})$ with the order carried over by the isomorphism Φ from $DG(\tau)$, then $(G_{B_{\mathcal{L}}}, B_{\mathcal{L}})$ has the strict order given by the homomorphism $\omega : G_{B_{\mathcal{L}}} \to \mathbb{R}$ defined by $\omega = \mu \circ \Phi$. Since Lebesgue measure is scaled by τ by the factor β, then $\mathcal{L}_*$ scales the associated state by the same factor. Thus for $v = (z_0, z_1, \ldots, z_{q-1}) \in \mathbb{Z}^q$,

$$\omega([v, n]) = \mu(\mathcal{L}_*^{-n}[\psi(v)]) = \beta^{-n} \sum_i z_i \mu(\mathcal{L}^i I(0,1)) = \beta^{-n} \sum_i z_i \beta^i.$$

□

We need the following result from [32]. Recall that a module is *cyclic* if it is singly generated.

Proposition 10.4. *([32, Prop. 7.3]) Let $\tau : I \to I$ be piecewise monotonic, with associated partition C. Assume that $DG(\tau)$ is cyclic, and that there exists $a \in \{0,1\}$ with an infinite orbit, such that*

$$\widehat{\tau}(C \setminus \{a\}) \subset C. \tag{30}$$

Then $DG(\tau) \cong \mathbb{Z}[t, t^{-1}]$ as abelian groups, with the action of $\mathcal{L}_*$ given by multiplication by t.

Proposition 10.5. *Let τ be the β-transformation. If 1 is not eventually periodic, then $DG(\tau)$ is isomorphic as a group to $\mathbb{Z}[t, t^{-1}]$, with the action of $\mathcal{L}_*$ given by multiplication by t. The unique state is given by $p \mapsto p(\beta)$, and $DG(\tau)$ has the strict order from this state.*

Proof. Each of the intervals $I(i/\beta, (i+1)/\beta)$ is mapped onto $I(0,1)$ by $\mathcal{L}$, and the jumps at partition points are all of size 1. From Theorem 2.12, it follows that $I(0,1)$ generates the module $DG(\tau)$. If a is a partition point other than 1, then $\widehat{\tau}(a) \subset \{0,1\}$, and 1 is not eventually periodic, so $DG(\tau) \cong \mathbb{Z}[t, t^{-1}]$ follows from Proposition 10.4. Lebesgue measure scales τ by the factor β, so takes $p(\mathcal{L})I(0,1)$ to $p(\beta)$. The remaining statements follow from Lemma 10.1). □

Note in Proposition 10.5 that when 1 is not eventually periodic, the range of the unique state is $\mathbb{Z}[\beta, \beta^{-1}]$. There will be infinitesimals iff β is algebraic, cf. Proposition 5.7. If β is transcendental, then $DG(\tau)$ is isomorphic as an ordered group to $\mathbb{Z}[\beta, \beta^{-1}]$, and the action of $\mathcal{L}_*$ on the latter is given by multiplication by β.

We now show that the dimension group $DG(F_\beta^\infty)$ defined in [17] is isomorphic to $DG(\tau_\beta)$.

Proposition 10.6. *If $\beta > 1$ and τ is the β-transformation, then the dimension groups $DG(\tau)$ and $K_0(F_\beta^\infty)$ are isomorphic as abelian groups, and are also order isomorphic if 1 is eventually periodic.*

Proof. If $x \in [0,1]$, then the *β-expansion* of x is $x = \sum_{i=1}^{\infty} \frac{\eta_i}{\beta^i}$, where for each k, η_k is the greatest integer in $\beta^k(x - \sum_{i=1}^{k-1} \frac{\eta_i}{\beta^i})$. The itinerary of 1 defined above Lemma 10.2 will be the sequence of coefficients $\eta_1\eta_2\eta_3\cdots$. If 1 is eventually periodic, Lemma 10.1 and [17, Prop. 3.5 and Lemma 6.2] imply that $DG(\tau_\beta)$ and $K_0(F_\beta^\infty)$ are simple dimension groups with unique states. Then Proposition 10.3 and [17, Lemma 4.9] show these groups are given by the same inductive limit, and have the same state, and thus the same strict order. If 1 is not eventually periodic, the desired group isomorphism follows from Proposition 10.5 and [17, Lemma 4.9, and proof of Thm. 6.1]. □

Remarks It is an open question whether $DG(\tau)$ and $K_0(F_\beta^\infty)$ are isomorphic as ordered abelian groups when 1 is not eventually periodic.

Let $S : [0,1) \to \{0, 1, \ldots, n-1\}^{\mathbb{N}}$ be the itinerary map for the β-transformation described above. Let X_β be the closure of the image of S in $\{0, 1, \ldots, n-1\}^{\mathbb{N}}$, where $n = [\beta]$, and let σ_β be the shift on X_β. Then σ_β is called the β-shift. We note that $S(I_0)$ will be dense in $S([0,1))$, so $X_\beta = \overline{S(I_0)}$.

Also denote by S the corresponding itinerary map on X with respect to the partition $I(0, 1\beta)$, $I(1/\beta, 2/\beta)$, ... Then the image of S on X_0 is the same as on

I_0, so $S(X_0)$ is dense in X_β. Since X_0 is dense in X and S is continuous on X, then $S(X) = X_\beta$, and S will be a semi-conjugacy from (X, σ) onto (X_β, σ_β).

If the orbit of 1 lands on 0, or $\tau 1 = 1$, then τ is Markov, and the itinerary map will be 1-1 on X, and thus will be a conjugacy from (X, σ) onto (X_β, σ_β). Since τ and σ are topologically exact, then (X, σ) is a shift of finite type [32, Prop. 8.5], and thus so is (X_β, σ_β).

If the orbit of 1 is finite but doesn't land on 0, and $\tau 1 \neq 1$, again τ is Markov, but now the itinerary map S with respect to the partition of X associated with $\{0, 1/\beta, \ldots, n/\beta, 1\}$ will not be 1-1. For example, if $x = \tau 1$, then x^+ and x^- will have the same itineraries. In this case the itinerary map is just a semi-conjugacy from the shift of finite type (X, σ) onto the sofic shift (X_β, σ_β).

Appendix A. Convergence of powers of the Perron-Frobenius operator

In this appendix $\tau : I \to I$ will be a piecewise monotonic map, with associated local homeomorphism $\sigma : X \to X$. We will investigate the convergence of the powers of the Perron-Frobenius operator associated with σ.

Standing Assumption Throughout this appendix, we will assume that there exists a non-atomic probability measure μ on X, with full support, scaled by σ by a factor $s > 1$ (cf. Definition 3.1).

The existence of such a scaling measure is equivalent to τ being conjugate to a piecewise linear map with slopes $\pm s$ with $s > 1$, cf. Propositions 3.3 and 3.6. By Proposition 4.3, such a measure will exist if τ is transitive and not essentially injective. Below, "a.e." will mean with respect to the measure μ.

We will begin with a summary of some results in a measure-theoretic context: convergence results are in the space $\mathcal{B}V$ (defined below) of functions of bounded variation in L^1, equalities of functions hold a.e., and partitions of X are a.e. These results are well known; mainly we will be following Rychlik [31]. An exposition of convergence results for the Perron-Frobenius operator can be found in [4].

For our application, it will be important to establish the corresponding results in a topological context, so that functions are continuous, convergence of sequences of functions is uniform, equalities of functions hold everywhere, and partitions of X are into clopen sets. For this purpose the key is working with the Perron-Frobenius operator associated with σ, rather than the one associated with τ. Since σ is a local homeomorphism, then the transfer operator $\mathcal{L}_\sigma$ (cf. Definition 2.9) maps $C(X)$ into $C(X)$, which is what allows us to transfer results to the topological context. For related results for the special case of β-transformations, see [33].

Let $P : L^1(X, \mu) \to L^1(X, \mu)$ be the map given by $P = (1/s)\mathcal{L}_\sigma$. Since σ scales μ by the factor s, then for $f \in \mathcal{L}^1(X, \mu)$, $\mu(\mathcal{L}f) = s\mu(f)$ (Lemma 3.2). Thus for $0 \le f$, $\|Pf\|_1 = \|f\|_1$, from which it follows that $\|P\|_1 = 1$. The map P is the Perron-Frobenius operator associated with σ, i.e., for all $f \in L^1(X, \mu)$ and all Borel sets $A \subset X$, it satisfies

$$\int_A (Pf)\, d\mu = \int_{\sigma^{-1}(A)} f\, d\mu \tag{31}$$

(cf. [31]).

For $f : X \to \mathbb{C}$, recall that the *variation* of f is

$$\operatorname{var} f = \sup \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|, \tag{32}$$

where the supremum is over all finite sequences $a_0 < a_1 < \cdots < a_n$ of points in X. We say f is of *bounded variation* if $\operatorname{var} f < \infty$. Every function f of bounded variation on X is the difference of increasing functions. (In fact, if a is the leftmost point of X, then we can write $f = g - h$ where $g(x) = \operatorname{var}_{[a,x]\cap X} f$ and $h = g - f$, cf. [28, §1.4]). It follows that f has left and right limits at each point, and that f is continuous except at a countable set of points.

Definition A.1. $\mathcal{B}V$ is the space of functions in $L^1(X, \mu)$ equal a.e. to a function of bounded variation, with the norm

$$\|f\|_{\mathcal{B}V} = \inf\{ \operatorname{var} g \mid g = f \text{ a.e.}\} + \|f\|_1 \tag{33}$$

where $\|f\|_1$ denotes the norm from $L^1(X, \mu)$.

Note that $\mathcal{B}V$ is a Banach space, cf. [31]. Each function of bounded variation is bounded, so $\mathcal{B}V \subset L^\infty(X, \mu)$. If $f \in \mathcal{B}V$ we write $\|f\|_\infty$ for the norm of f as a function in $L^\infty(X, \mu)$.

Lemma A.2. *If $f \in \mathcal{B}V$, then*

$$\|f\|_\infty \le \|f\|_{\mathcal{B}V}. \tag{34}$$

Proof. Let f have bounded variation. Since $\|f\|_1 \ge \inf_{x\in X} |f(x)|$, for each $\epsilon > 0$, there is some $y \in X$ such that $|f(y)| \le \|f\|_1 + \epsilon$. Then for each $x \in X$,

$$|f(x)| \le |f(x) - f(y)| + |f(y)| \le \operatorname{var} f + \|f\|_1 + \epsilon, \tag{35}$$

so $\|f\|_\infty \le \operatorname{var} f + \|f\|_1$. As f varies in its equivalence class in L^1, $\|f\|_\infty$ doesn't change, and the infimum of the numbers $\operatorname{var} f + \|f\|_1$ is $\|f\|_{\mathcal{B}V}$. The inequality in the lemma follows. □

Proposition A.3. *(Rychlik* [31, Thm. 1]*) P maps $\mathcal{B}V$ into $\mathcal{B}V$. If P is considered as an operator on $\mathcal{B}V$, the set λ_1, λ_2, …, λ_q of points in the spectrum of P with modulus 1 is finite and non-empty. These points are all roots of unity, and are simple poles of the resolvent. There is a number $r < 1$ such that all other points in the spectrum are contained in the disk around* 0 *of radius r. If for each i, Q_i is the spectral projection corresponding to λ_i, then $P = \sum_i \lambda_i Q_i + R$, where each Q_i is a finite rank projection whose range is the eigenspace of P for eigenvalue λ_i, with $Q_iQ_j = 0$ for $i \ne j$, $Q_iR = RQ_i = 0$ for each i, and with the spectral radius of R less than 1.*

Let N be the least positive integer in Proposition A.3 such that $\lambda_i^N = 1$ for all i, and let $Q = \sum_i Q_i$. Note that Q is a projection onto the space spanned by all eigenvectors corresponding to eigenvalues of modulus 1, or equivalently, onto the space of fixed points of P^N.

Lemma A.4. *If $f \in C(X) \cap \mathcal{B}V$, then f has bounded variation.*

Proof. Choose g of bounded variation, such that $f = g$ a.e. Let $x_1 < x_2 < \ldots < x_k$ be any points in X. Since μ has full support, in any open interval around a point $x \in X$, there exists a point t such that $g(t) = f(t)$. Given $\epsilon > 0$, choose points

$t_1 < t_2 < \ldots < t_k$ such that $\sum_i |f(x_i) - f(t_i)| < \epsilon/2k$ and such that $g(t_i) = f(t_i)$ for all i. Then

$$\sum_i |g(t_i) - g(t_{i+1})| = \sum_i |f(t_i) - f(t_{i+1})| \geq \sum_i |f(x_i) - f(x_{i+1})| - \epsilon. \tag{36}$$

It follows that $\operatorname{var} g \geq \operatorname{var} f$, so f has bounded variation. □

Corollary A.5. *For all $f \in \mathcal{B}V$,*

$$Qf = \lim_k P^{Nk} f, \tag{37}$$

where convergence is in $\mathcal{B}V$. If $f \in C(X) \cap \mathcal{B}V$, then $P^{Nk} f$ converges uniformly to a function in $C(X) \cap \mathcal{B}V$, and so Qf is equal a.e. to a function in $C(X) \cap \mathcal{B}V$.

Proof. The first statement is an immediate consequence of Proposition A.3. Recall that $P = (1/s)\mathcal{L}_\sigma$ maps $C(X)$ into $C(X)$ (as noted after Lemma 3.2). If $f \in C(X) \cap \mathcal{B}V$, since the $\mathcal{B}V$ norm dominates the L^∞ norm (Lemma A.2), then $\|P^{Nk} f - Qf\|_\infty \to 0$. Hence $\{P^{Nk} f\}_{k=1}^\infty$ is a sequence of continuous functions, and is Cauchy with respect to the supremum norm. Let g be the uniform limit of this sequence. Then g is continuous, and $g = Qf$ a.e. By Lemma A.4, g has bounded variation. □

If A, B are subsets of X, we write $A = B$ a.e. if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ has measure zero with respect to μ. Note that all equalities of functions in Theorem A.6 are a.e. In Theorem A.6, Q denotes the map defined in Proposition A.3, and N is the integer defined in the remarks following that proposition.

Theorem A.6. *(Rychlik* [31, Thms. 3, 4]*) Then there are nonnegative functions $\phi_1, \ldots, \phi_q \in \mathcal{B}V$ and $\psi_1, \ldots, \psi_q \in L^\infty$ such that*

(i) *Q has a unique extension to a bounded operator on L^1, and Q maps L^1 into $\mathcal{B}V$.*

(ii) *For every $f \in L^1$,*

$$Qf = \sum_{i=1}^q \mu(\psi_i f)\phi_i.$$

(iii) *$P^N \phi_i = \phi_i$, $\psi_i \circ \sigma^N = \psi_i$ for $i = 1, \ldots, q$.*

(iv) *$\mu(\phi_i \psi_j) = \delta_{ij}$, $\min(\psi_i, \psi_j) = 0 = \min(\phi_i, \phi_j)$ for $i \neq j$, $\mu(\phi_i) = 1$ for $i = 1, \ldots, q$.*

(v) *There exist measurable sets $C_1, \ldots, C_q \subset X$ such that $\psi_i = \chi_{C_i}$ a.e. for $i = 1, \ldots, q$, and $X = \cup_{i=1}^q C_i$ a.e.*

(vi) *There exists a permutation ω of the set $\{1, \ldots, q\}$ such that*

$$P\phi_i = \phi_{\omega(i)}, \qquad \psi_{\omega(i)} \circ \sigma = \psi_i \quad \text{for } i = 1, \ldots, q, \tag{38}$$

and such that ω^N is the identity. If ω is a cycle, then $q = N$.

Proof. This result is contained in [31, Thms. 3, 4], except for the last statement in (vi). Suppose ω is a q-cycle. By (iv), ϕ_1, ϕ_2, …, ϕ_q are distinct, so by (vi), ϕ_1, $P\phi_1$, …, $P^{q-1}\phi_1$ are distinct. By (38), $P^N \phi_1 = \phi_1$, which implies that $N \geq q$.

Since ω is a q-cycle, $P^q \phi_i = \phi_i$ for all i. By (ii), the range of Q is spanned by ϕ_1, …, ϕ_q, so $P^q Q = Q$. Let λ be an eigenvalue of modulus 1 for P, with eigenvector ψ. By construction, Q fixes each such eigenvector, so $P^q Q\psi = Q\psi$ implies that $P^q \psi = \lambda^q \psi = \psi$. Hence $\lambda^q = 1$ for each eigenvalue of P of modulus 1, so by the definition of N, $q \geq N$. Thus $N = q$. □

By definition, the support of a function $f \in C(X)$ is the closure of the set $\{x \mid f(x) \neq 0\}$. If $f \in C(X, \mathbb{Z})$, then $\{x \mid f(x) \neq 0\}$ is already closed, so $\operatorname{supp} f$ consists of the points where f is not zero. From the definition of $\mathcal{L}_\sigma$,

$$0 \leq f \in C(X) \implies \operatorname{supp}(\mathcal{L}_\sigma f) = \sigma(\operatorname{supp} f). \tag{39}$$

We are now going to strengthen the conclusions of Theorem A.6 by showing that each ϕ_i can be chosen to be continuous, with clopen support. Recall that μ is a measure of full support on X, scaled by σ by the factor s.

Corollary A.7. *Each ϕ_i in Theorem A.6 can be chosen to be continuous and of bounded variation. For such a choice of ϕ_i, if $f \in C(X) \cap \mathcal{B}V$ with $\operatorname{supp} f \subset \{x \mid \phi_i(x) > 0\}$, then $P^{Nk} f$ converges uniformly to $\mu(f)\phi_i$.*

Proof. By definition of $\mathcal{B}V$, each ϕ_i is equal a.e. to a function of bounded variation, so without loss of generality we may assume ϕ_i has bounded variation on X. Then ϕ_i is continuous except on a countable set of points (and thus a.e., since μ is non-atomic). If ϕ_i were zero at each point of continuity, then it would be zero a.e., which would contradict $\mu(\psi_i \phi_i) = 1$. Thus there is an open subset V of X such that $\phi_i > 0$ on V.

Now let f be any non-negative continuous function of bounded variation, not identically zero, such that the support of f is contained in $\{x \mid \phi_i(x) > 0\}$. By Theorem A.6 (iv) and (v), $\{x \mid \phi_i(x) > 0\} \subset C_i$ a.e., so the support of f is contained in C_i a.e. By Theorem A.6 (ii), $Qf = \mu(f)\phi_i$, and since μ has full support, $\mu(f) > 0$.

By Proposition A.5, $P^{Nk} f$ converges uniformly to $\mu(f)\phi_i'$, where ϕ_i' in $C(X) \cap \mathcal{B}V$ is equal a.e. to Qf. Thus $\phi_i = \phi_i'$ a.e., so by replacing ϕ_i by ϕ_i', we may choose ϕ_i to be continuous.

We have proven that $P^{Nk} f$ converges uniformly to $\mu(f)\phi_i$ for $0 \leq f \in C(X) \cap \mathcal{B}V$ with $\operatorname{supp} f \subset \{x \mid \phi_i(x) > 0\}$. If f is not necessarily non-negative, the same convergence result follows by writing $f = f^+ - f^-$, with $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. ☐

Hereafter we assume each ϕ_i in Theorem A.6 has been chosen to be continuous and of bounded variation. Note that with such a choice for each ϕ_i, the equalities $P^N \phi_i = \phi_i$ and $P\phi_i = \phi_{\omega(i)}$ in Theorem A.6 (vi) are equalities of continuous functions, so will hold everywhere, not just a.e.

Proposition A.8. *For each i, $\{x \mid \phi_i(x) > 0\}$ is clopen (and thus equals the support of ϕ_i).*

Proof. Let $g : I_0 \to \mathbb{R}$ be defined by $g(x) = \phi_i(\pi^{-1}(x))$, where $\pi : X \to I$ is the collapse map. Since $\phi_i \in C(X) \cap \mathcal{B}V$, then ϕ_i is of bounded variation (Lemma A.4), so g will be the difference of increasing functions on I_0. It follows that g can be extended to the difference of increasing functions on I, i.e., to a function of bounded variation on I, which we also label as g.

Let m be the measure on I scaled by τ that corresponds to μ, cf. Proposition 3.3. Recall that there is a conjugacy of (I, τ) onto a piecewise linear map with slopes $\pm s$, which carries m to Lebesgue measure (Proposition 3.6). Therefore we may assume, without loss of generality, that τ is piecewise linear with slopes $\pm s$, and that m is Lebesgue measure.

Now let $V = \{x \mid \phi_i(x) > 0\}$. By continuity of ϕ_i, V is open. Let $W = \pi(V \cap X_0)$. Since $\pi : X_0 \to I_0$ is a homeomorphism (Proposition 2.3), then W is an open subset of I_0 on which g is positive.

Since $\tau'(x) = 1/s$ a.e, then $(1/s)L_\tau$ is the Perron-Frobenius operator $P_\tau : L^1(I,m) \to L^1(I,m)$, cf. [4, eqn. (4.3.4)]. Since $((1/s)\mathcal{L}_\sigma)^N\phi_i = P_\sigma^N \phi_i = \phi_i$, then $P_\tau^N g = ((1/s)\mathcal{L}_\tau)^N g = g$ on I_0, so $(P_\tau)^N g = g$ a.e. Since $(P_\tau)^N = P_{\tau^N}$, then $P_{\tau^N} g = g$ a.e.

Choose $g_1 : I \to I$ lower semi-continuous so that $g = g_1$ a.e., cf. [4, Lemma 8.1.1]. Then g_1 is a fixed point of the Perron-Frobenius operator P_{τ^N}. Since g_1 is lower semi-continuous, by [4, Thm. 8.2.3] there is a constant $\alpha > 0$ such that $g_1(x) \geq \alpha$ for all x such that $g_1(x) > 0$. In particular, $g_1(x) \geq \alpha$ a.e. on W. It follows that $\phi_i(x) \geq \alpha$ a.e. on the set $V \cap X_0$, which is dense in V. By continuity of ϕ_i, and density of X_0, $\phi_i \geq \alpha$ on V. Thus $V = \{x \in X \mid \phi_i(x) \geq \alpha\}$, so V is clopen. □

Corollary A.9. *With the notation of Theorem A.6, let* $X_i = \operatorname{supp}\phi_i$ *for* $1 \leq i \leq q$. *For each* i, $\sigma(X_i) = X_{\omega(i)}$, *and* σ^N *leaves* X_i *invariant and is topologically exact on* X_i. *The sets* $X_1, \ldots, X_q$ *are disjoint.*

Proof. Since $\min(\phi_i, \phi_j) = 0$ for $i \neq j$, then $X_i \cap X_j = \emptyset$ for $i \neq j$. By (39), the support of $\phi_{\omega(i)} = (1/s)\mathcal{L}_\sigma\phi_i$ is

$$\operatorname{supp}(\mathcal{L}_\sigma\phi_i) = \sigma(\operatorname{supp}\phi_i) = \sigma(X_i), \tag{40}$$

so $\sigma(X_i) = X_{\omega(i)}$. Since ω^N is the identity permutation, then $\sigma^N X_i = X_i$ for all i.

Fix i and let V be any open subset of X_i. Now choose $\epsilon > 0$ so that $\phi_i > \epsilon$ on X_i. Choose a nonnegative continuous function f of bounded variation, not identically zero, such that f is zero outside V. By Corollary A.7, $s^{-Nk}\mathcal{L}^{Nk}f$ converges to $\alpha\phi_i$, where $\alpha = \mu(f) \neq 0$.

Choose $k > 0$ such that $\|s^{-Nk}\mathcal{L}^{Nk}f - \alpha\phi_i\|_\infty < \alpha\epsilon/2$. It follows that $\mathcal{L}^{Nk}f > 0$ on all of X_i. Since $\mathcal{L}^{Nk}f$ is zero outside $\sigma^{Nk}V$, then $\sigma^{Nk}(V) = X_i$. This completes the proof that σ^N is exact on X_i. □

Remarks (i) Recall that our standing assumption in this appendix requires the existence of a measure scaled by τ by a factor $s > 1$. If $s = 1$, then the results above can fail. (See the example after Corollary 4.7.)

(ii) Simple examples show that the union of the sets X_i need not be all of X. However, this will be the case if τ is transitive, cf. Theorem 4.5.

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