Fundamentals

- Various integration techniques: u-substitution, integration by parts, partial fractions.
- Trigonometric identities: $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\cos(x + y) = \cos x \cos y - \sin x \sin y$, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, $\cos^2 x = (1 + \cos 2x)/2$, $\sin^2 x = (1 - \cos 2x)/2$.
- Limits and improper integrals.

Chapter 2: First Order Differential Equations

- Know how to distinguish between linear and non-linear equations.
- <u>Section 2.1</u>: Linear equations can always be written in the form $y' + p(t)y = g(t)$, and they are solved with an integrating factor $\mu(t) = e^{\int p(t)dt}$:

$$[\mu(t)y]' = \mu(t)y' + \mu(t)p(t)y = \mu(t)g(t).$$

- <u>Section 2.2</u>: If an equation is non-linear, it may be separable: it can be written in the form $f(x)dx = g(y)dy$. In this case, integrate both sides and solve for $y(x)$, if possible. Otherwise, leave it as an implicit equation in x and y.
- **Homogeneous** equations (page 49) can be solved with the change of variables $v = y/x$. The resulting equation will be *separable* in v.
- <u>Section 2.3</u>: Important applications include mixing problems, compound interest, population growth, and radioactive decay.
- <u>Section 2.4</u>: Theorems 2.4.1 and 2.4.2 describe the conditions under which there is a solution to a particular differential equation. For linear equations, $p(t)$ and $g(t)$ have to be continuous on an interval containing t_0. For non-linear equations $y' = f(t, y)$, both f and $\partial f/\partial y$ have to be continuous on a rectangle containing (t_0, y_0).
- **Bernoulli** equations (page 77) are of the form $y' + p(t)y = q(t)y^n$ where $n = 0, 1, 2, \ldots$; when $n = 0$ or 1, the equation is already linear. For $n \geq 2$, use the change of variables $v = y^{1-n}$, and the new equation is linear in v.
- <u>Section 2.5</u>: Autonomous equations are of the form $y' = f(y)$. Review the method for sketching solutions using the equilibrium points and phase line. Exponential growth ($y' = ry$) and logistic growth ($y' = (r - ay)y$) are examples of autonomous equations.
- <u>Section 2.6</u>: In order to determine exactness, first write the equation in the form $M(x, y) + N(x, y)y' = 0$ or $M(x, y)dx + N(x, y)dy = 0$; it is exact if $M_y = N_x$. If it is not exact, an integrating factor might help: if $\frac{M_y - N_x}{N}$ is *only* a function of x, then $\mu'(x) = \left(\frac{M_y - N_x}{N}\right)\mu(x)$. If $\frac{N_x - M_y}{M}$ is *only* a function of y, then $\mu'(y) = \left(\frac{N_x - M_y}{M}\right)\mu(y)$.
- <u>Section 2.9</u>: Working problems in this section is an excellent way to review the methods for solving 1st order differential equation. Also covered in this section are **Riccati** equations and some special 2nd order differential equations: $y'' = f(t, y')$ (to solve set $v = y', v' = y''$) and $y'' = f(y, y')$ (set $v = y'(t)$, then $dv/dt = v(dv/dy)$).

Chapter 3 & 4: Higher Order Differential Equations

In these two chapters, we are primarily interested in linear equations with constant coefficients. Due to the similarities of the methods covered in both chapters, I will group by topic rather than text section.

- First we learn to solve **homogeneous** equations; they look like

$$\begin{aligned} &ay'' + by' + cy = 0, && \text{(2nd order)} \\ &ay^{(3)} + by'' + cy' + dy = 0, && \text{(3rd order)} \\ &a_0y^{(4)} + a_1y^{(3)} + a_2y'' + a_3y' + a_4y = 0, && \text{(4th order)} \end{aligned}$$

  etc. In order to find the solutions, we find the roots to the characteristic equation:

$$\begin{aligned} &ar^2 + br + c = 0, && \text{(2nd degree)} \\ &ar^3 + br^2 + cr + d = 0, && \text{(3rd degree)} \\ &a_0r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0, && \text{(4th degree)} \end{aligned}$$

  etc.

  1. If r_0 is a real root to the characteristic equation, then e^{rot} is a solution to the differential equation. If r_0 is a root twice then te^{rot} is also a solution; if r_0 is a root 3 times, then t^2e^{rot} is also a solution, etc.
  2. If $\lambda + i\mu$ is a complex root of the characteristic equation, then $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$ are solutions to the differential equation. If $\lambda + i\mu$ is a root twice then $te^{\lambda t}\cos\mu t$ and $te^{\lambda t}\sin\mu t$ are also solutions; if $\lambda + i\mu$ is a root 3 times then $t^2e^{\lambda t}\cos\mu t$ and $t^2e^{\lambda t}\sin\mu t$ are also solutions, etc.
  3. Take all your individual solutions $y_1, y_2, \ldots, y_n$ from above and form the general solution: $y_c(t) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ where $c_1, c_2, \ldots, c_n$ are constants. The general solution for a 2nd order equation will be built out of a fundamental set of 2 solutions, the general solution for a 3rd order equation will be built out of a fundamental set of 3 solutions, etc.
  4. Understand the relationships between a fundamental set of solutions, linear independence, and the Wronskian: if a set of solutions is linearly independent, it is a fundamental set of solutions (and vice versa). If the Wronskian of a set of solutions is non-zero, it is a fundamental set of solutions (and vice versa). Finally, if a set of solutions is linearly independent, their Wronskian is non-zero (and vice versa).

- Know **Abel's** Theorem, both for 2nd order (page 155) and higher order differential equations (page 223, #20).

- **Euler's** equations (page 166) are of the form $t^2y'' + \alpha ty' + \beta y = 0$. They don't have constant coefficients, but the substitution $x = \ln t$ results in an equation with constant coefficients.

- The method of **reduction of order** (page 171) can be used to find a 2nd solution to a 2nd order equation once we already know one solution.

- In order to solve **non-homogeneous** equations, we must use either the method of **undetermined coefficients** or the method of **variation of parameters** to find a particular solution $Y(t)$. Then the general solution to the non-homogeneous equation is $y(t) = y_c(t) + Y(t)$ where $y_c(t)$ is the solution to the homogeneous equation. Review both methods carefully because there still seems to be a lot of confusion concerning each method.

- Sections 3.8 & 3.9: covered mechanical vibrations, both free and forced. For mass m, damping coefficient γ, spring constant k, and external force $F(t)$ the behavior of the spring-mass system is described by the 2nd order differential equation $mu'' + \gamma u' + ku = F(t)$. If $F(t) = 0$ the system is a free vibration; if $F(t) \neq 0$ the system is a forced vibration. Know the difference between un-damped, over-damped and critically damped systems. If under-damped, know the quasi-period. Be able to determine the natural spring frequency and understand its connection with resonance.

Chapter 6: The Laplace Transform

- Know the definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$.

- Be able to calculate the transform of all the basic functions, given the table on page 319.

- Know how to compute inverse transforms; partial fractions are a useful tool.

- Know how to transform derivatives of functions and linear differential equations.

- Understand the step function $u_c(t)$ and the unit impulse function $\delta(t - c)$. Laplace transforms are especially useful when solving differential equations where the forcing function involves step and/or impulse functions.

- **Convolution Integral**: We don't have a formula for $\mathcal{L}\{f(t)g(t)\}$, but we *do* have a formula for $\mathcal{L}^{-1}\{F(s)G(s)\}$ where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$$

  A simple change of variables shows that $f * g = g * f$, so it doesn't matter which function you decide to shift: $\int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t g(t - \tau)f(\tau)d\tau$.

Chapter 7: Systems of First Order Linear Equations

- Know the fundamentals of matrix algebra.

- Know how to find the inverse of a matrix.

- Be able to find the solution to a set of linear algebraic equations.

- Understand what it means for vectors to be linearly independent.

- Know how to find the eigenvalues and eigenvectors of a matrix.

- Sketch the direction field for a 2×2 system of linear differential equations.

- Find the general solution for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

1. If the eigenvalues are distinct, then each solution is of the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{\lambda t}$ where λ is an eigenvalue with eigenvector $\boldsymbol{\xi}$ ($\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$).
2. If λ is a repeated eigenvalue with eigenvector $\boldsymbol{\xi}$, then one solution is $\mathbf{x}_1(t) = \boldsymbol{\xi} e^{\lambda t}$, while another solution is $\mathbf{x}_2(t) = \boldsymbol{\xi} t e^{\lambda t} + \boldsymbol{\eta} e^{\lambda t}$ where $\boldsymbol{\eta}$ is a *generalized* eigenvector: $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.
3. If $\lambda = a + bi$ is a complex eigenvalue, then its eigenvector $\boldsymbol{\xi}$ will also be complex (when $\mathbf{A}$ is a real matrix). In this case, $\boldsymbol{\xi} e^{\lambda t} = \boldsymbol{\xi} e^{(a+bi)t} = \boldsymbol{\xi} e^{at}(\cos bt + i \sin bt)$. Multiply this out and write it in the form $\boldsymbol{\xi} e^{\lambda t} = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$, then one solution is $\mathbf{x}_1(t)$ and another is $\mathbf{x}_2(t)$.

- Know how to find a fundamental matrix $\boldsymbol{\Psi}(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$. The fundamental matrix $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)$ has the nice property that $\boldsymbol{\Phi}(t_0) = \mathbf{I}$ and so $\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}^0$ satisfies the initial condition $\mathbf{x}(t) = \mathbf{x}^0$.
- For non-homogeneous systems $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, find a particular solution $\mathbf{X}(t)$ using undetermined coefficients or variation of parameters. Definitely review this section because a lot of people had trouble with the homework.