APMA 213, Summer 2005
First Midterm Exam

Name: SOLUTIONS Student ID: ____________

Please sign the pledge: On my honor as a student I have neither given nor received aid on this examination.

Signature: ____________

Directions: Check that your test has 8 pages, including this one and the blank one on the bottom. Please answer all questions and show your work. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** Closed book, closed notes, no calculators.

1. (10 points) ____________
2. (5 points) ____________
3. (5 points) ____________
4. (5 points) ____________
5. (5 points) ____________
6. (10 points) ____________

Total (out of 40): ____________

1. (2 pts each) Answer each of the following questions by circling TRUE or FALSE. You do not need to justify you answer and no partial credit will be given.

(a) TRUE FALSE

A differential equation is called *linear* if it is an expression in terms of an independent variable, a function of that variable, and the first derivative of that function.

(b) TRUE FALSE

An equation of the form $M(x) + N(y)y' = 0$ is always separable.

(c) TRUE FALSE

An equation of the form $M(x) + N(y)y' = 0$ is always exact.

(d) TRUE FALSE

If $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation $ay'' + by' + cy = 0$ and $W(y_1, y_2) \neq 0$ (i.e. the Wronskian of y_1 and y_2 is not zero for all t), then $c_1y_1(t) + c_2y_2(t)$, where c_1 and c_2 are arbitrary constants, is a general solution for that homogeneous equation.

(e) TRUE FALSE

Equation $y'' + p(t)y' + q(t)y = g(t)$ with initial conditions $y(t_0) = y_0$, $y'(t_0) = y'_0$, always has a solution.

2. (5 pts) Find a general solution to the equation

$$t^2y' + 3ty = \frac{\sin t}{t}.$$

Rewrite as

$$(*) \qquad y' + \frac{3}{t}y = \frac{\sin t}{t^3}$$

and use integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3.$$

Multiplying $(*)$ by $\mu(t)$ gives

$$t^3y' + t^2y = \sin t$$

$$\iff \qquad (t^3y)' = \sin t \text{ . Integrate both sides:}$$

$$t^3y = -\cos t + C$$

$$\boxed{y = \frac{-\cos t + C}{t^3}}$$

3. (5 pts) Find a general solution to the equation

$$\underbrace{(x^2 + y)}_{M}dx + \underbrace{(x + e^y)}_{N}dy = 0.$$

Since $M_y = 1 = N_x$, this is an exact eq'n.

Thus there is a $\psi(x,y)$ such that

$\psi_x = M$, $\psi_y = N$. To find ψ,

first compute

$$\psi = \int M\,dx = \int (x^2 + y)\,dx = \frac{x^3}{3} + xy + \underbrace{h(y)}_{\text{some function of } y \text{ only.}}$$

To find $h(y)$, compute

$$\psi_y = x + h'(y) = x + e^y$$

$$\Rightarrow \quad h'(y) = e^y \quad \Rightarrow \quad h(y) = e^y$$

Thus $\psi(x,y) = \frac{x^3}{3} + xy + e^y$ and the solution to the original equation is

$$\boxed{\frac{x^3}{3} + xy + e^y = c}$$

4. (5 pts) Consider the autonomous equation

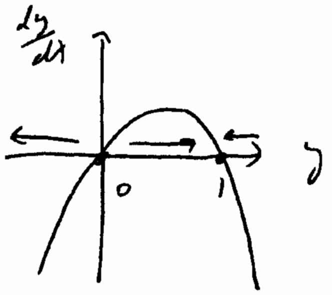
$$\frac{dy}{dx} = y(1-y)$$

with the initial condition $y(0) = y_0$ which could be any real number. Determine the equilibrium points (constant solutions) of the equation. Sketch the graph of dy/dx versus y and use it to classify the equilibrium points as asymptotically stable, unstable, or semistable.

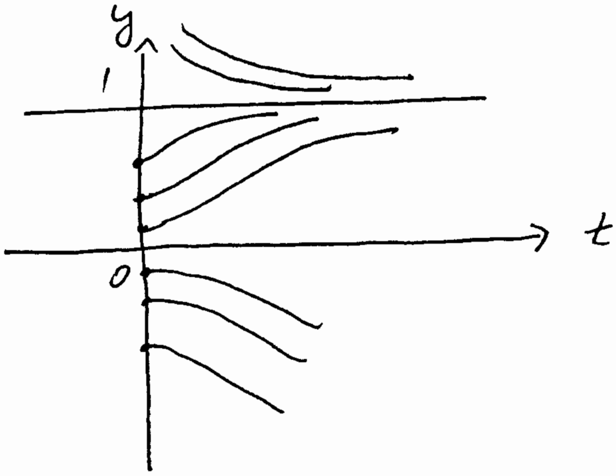
Equilibrium solutions are

$y = 0$, $y = 1$ (i.e. when $\frac{dy}{dx} = 0$)

Graph of dy/dx vs. y is



which allows us to examine if the solutions increase or decrease depending on the initial condition, i.e. have



Thus $\underline{y=0}$ is <u>unstable</u> and

$\underline{y=1}$ is <u>stable</u>.

5. (5 pts) Suppose a mass of 10 kg stretches a spring 5 cm. If the spring is extended another 2 cm and released, the mass is set in motion with initial velocity of 1 cm/sec. Determine the position of the mass at any time as well as the frequency of the motion.

Have $k = \text{weight}/\text{stretch} = \dfrac{10\,\text{kg} \cdot 980\,\text{cm}/\text{s}^2}{5\,\text{cm}}$

$= 1960$

$m = 10$

Since there is no damping or external force, the equation of motion is

$$mu'' + ku = 0$$

or $$10u'' + 1960u = 0$$

or $$u'' + 196u = 0$$

$$\boxed{\text{Frequency is } \omega_0 = \sqrt{k/m} = \sqrt{196} = 14}$$

The characteristic equation is

$r^2 + 196 = 0$ with solutions

$r_{1,2} = \pm 14i$. So a general solution is

$$u = c_1\cos(14t) + c_2\sin(14t)$$

Plugging in initial conditions

$u(0) = 2$, $u'(0) = 1$ gives the particular solution

$$\boxed{u = 2\cos(14t) + \frac{1}{14}\sin(14t)}$$

6. (10 pts) Find a general solution to the equation

$$y'' - 2y' - 3y = 3te^{2t}.$$

First solve the homogeneous problem

$$y'' - 2y' - 3y = 0 .$$

The characteristic equation is

$$r^2 - 2r - 3 = 0 \quad ; \quad \text{with roots}$$

$$r_1 = -1 \quad , \quad r_2 = 3 .$$

So the complementary solution is

$$y_c = c_1 e^{-t} + c_2 e^{+3t} .$$

Now find a particular solution. A guess is

$$y = (At+B) e^{2t}$$

(Since this guess does not match either of the solutions in y_c, we don't have to modify it)

Then

$$y' = Ae^{2t} + 2(At+B)e^{2t} = (A + 2At + 2B)e^{2t}$$

$$y'' = 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t}$$

$$= (4A + 4At + 4B)e^{2t}$$

Plugging this into the original equation gives

$$(4A + 4At + 4B)\cancel{e^{2t}} - 2(A + 2At + 2B)\cancel{e^{2t}} - 3(At + B)\cancel{e^{2t}}$$

$$= 3t\cancel{e^{2t}}$$

or

$$-3At + 2A - 3B = 3t$$

$$\Rightarrow \quad \left.\begin{array}{r} -3A = 3 \\ 2A - 3B = 0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} A = -1, \\ B = -2/3 \end{array}$$

So a general solution is

$$y = y_c + Y$$

$$= c_1 e^{-t} + c_2 e^{+3t} + (-1 \cdot t - 2/3) e^{2t}$$

$$= \boxed{c_1 e^{-t} + c_2 e^{+3t} - (t + 2/3) e^{2t}}$$