APMA 213, Summer 2005
Second Midterm Exam

Name: SOLUTIONS Student ID: ______________________

Please sign the pledge: On my honor as a student I have neither given nor received aid on this examination.

Signature: ______________________________

Directions: Check that your test has 9 pages, including this one and the blank one on the bottom. The next to last page should be a table of Laplace transforms. Please answer all questions and show your work. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** Closed book, closed notes, no calculators.

1. (10 points) ____________
2. (10 points) ____________
3. (10 points) ____________
4. (10 points) ____________
5. (10 points) ____________
6. (10 points) ____________

Total (out of 60): ____________

1. (2 pts each) Answer each of the following questions by circling TRUE or FALSE. You do not need to justify you answer and no partial credit will be given.

(a) $\boxed{\text{TRUE}}$ FALSE

The method of variation of parameters is a method for finding a particular solution to a non-homogeneous linear differential equation.

(b) TRUE $\boxed{\text{FALSE}}$

If $f(t)$ and $g(t)$ are continuous for all t, $f(t) \geq g(t) \geq 0$ for all t, and $\int_0^\infty g(t)\,dt$ converges, then $\int_0^\infty f(t)\,dt$ converges as well.

(c) TRUE $\boxed{\text{FALSE}}$

The value of the Dirac delta function $\delta(t)$ at $t = 0$ is 1.

(d) TRUE $\boxed{\text{FALSE}}$

Let $\mathcal{L}$ denote the Laplace transform. Then

$$\mathcal{L}(f(t)g(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t)).$$

(e) $\boxed{\text{TRUE}}$ FALSE

Let $\mathcal{L}$ denote the Laplace transform and let $*$ denote the convolution product. Then

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t)).$$

2. (a) (5 pts) Find a general solution to the homogeneous equation

$$y''' - 3y'' + 4y' - 2y = 0.$$

Characteristic equation is

$r^3 - 3r^2 + 4r - 2 = 0$, whose one

solution is $r = 1$ by guessing. By long division, the equation factors as

$$(r-1)(r^2 - 2r + 2) = 0 .$$

The quadratic $r^2 - 2r + 2$ has roots $r = 1 \pm i$.
So the general solution is

$$y(t) = c_1 e^t + c_2 e^t \cos t + c_3 e^t \sin t$$
$$= e^t (c_1 + c_2 \cos t + e^t \sin t)$$

(b) (5 pts) Using part (a) and the method of undetermined coefficients, determine a suitable form for $Y(t)$, a particular solution to the non-homogeneous equation

$$y''' - 3y'' + 4y' - 2y = e^t + e^t \sin t.$$

Please DO NOT find the undetermined coefficients.

The particular solution is $Y(t) = Y_1(t) + Y_2(t)$,
where

$$Y_1(t) = A t e^t$$

$$Y_2(t) = B t e^t \cos t + C t e^t \sin t$$
$$= t e^t (B \cos t + C \sin t)$$

3. Functions $y_1(t) = 1$, $y_2(t) = e^t$, $y_3(t) = \cos t$, and $y_4(t) = \sin t$ are solutions to the equation

$$y^{(4)} - y''' + y'' - y' = 0.$$

The Wronskian of these solutions is $W(t) = 2e^t$, while $W_2(t) = 1$.

(a) (5 pts) Do y_1, y_2, y_3, and y_4 form a fundamental set of solutions? Are they linearly independent? Justify your answers.

We have $W(t) = 2e^t \neq 0$ and

$W(t) \neq 0 \iff$ solutions form a fundamental set

$\iff$ solutions are linearly independent

So answer to both questions is yes.

(b) (5 pts) Equation

$$y^{(4)} - y''' + y'' - y' = e^t \tan t.$$

has a particular solution of the form

$$Y(t) = u_1(t) + u_2(t)e^t + u_3(t)\cos t + u_4(t)\sin t.$$

Find $u_2(t)$.

$$u_2(t) = \int_{t_0}^{t} \frac{g(s) W_2(s)}{W(s)}\, ds = \int_{t_0}^{t} \frac{e^s \tan s \cdot 1}{2e^s}\, ds$$

$$= \int_{t_0}^{t} \tan s\, ds = \ln(\sec s)\Big|_{t_0}^{t}$$

$$= \ln(\sec t) - \ln(\sec t_0)$$

Choose $t_0 = 0$ so that $\ln(\sec t_0) = \ln 1 = 0$

and thus

$$\boxed{u_2(t) = \ln(\sec t)}$$

4. Find the Laplace transform of the following functions.

(a) (5 pts) $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ 1, & 2 \le t < 3, \\ 0, & 3 \le t. \end{cases}$

$$f(t) = u_2(t) - u_3(t) \qquad \text{So}$$

$$\mathcal{L}(f(t)) = \mathcal{L}(u_2(t) - u_3(t)) = \mathcal{L}(u_2(t)) - \mathcal{L}(u_3(t))$$

$$= \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} = \boxed{\frac{e^{-2s} - e^{-3s}}{s}}$$

(b) (5 pts) $f(t) = \int_0^t (t-\tau)^2 \cos\tau \, d\tau.$

$$f(t) = \int_0^t (t-\tau)^2 \cos\tau \, d\tau = t^2 * \cos t$$

So

$$\mathcal{L}(f(t)) = \mathcal{L}(t^2 * \cos t) = \mathcal{L}(t^2)\,\mathcal{L}(\cos t)$$

$$= \frac{6}{s^3} \cdot \frac{s}{s^2+1} = \boxed{\frac{6}{s^2(s^2+1)}}$$

Book table lines 3 and 6
(or last sheet table lines 3 and 5)

5. Find the inverse Laplace transform of the following functions.

(a) (5 pts) $F(s) = \dfrac{s+4}{s^2+2s+2}$

$$\frac{s+4}{s^2+2s+2} = \frac{s+4}{(s+1)^2+1} = \frac{s+1+3}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}$$

So

$$\mathcal{L}^{-1}\left(\frac{s+4}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) + 3\,\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= \boxed{e^{-t}\cos t + 3e^{-t}\sin t}$$

book table lines 10 and 9
(or last sheet table lines 9 and 8)

(b) (5 pts) $F(s) = \dfrac{4e^{-3s}}{(s+3)(s-1)}$.

$$\frac{4e^{-3s}}{(s+3)(s-1)} = e^{-3s}\frac{4}{(s+3)(s-1)} = e^{-3s}\left(\frac{-1}{s+3} + \frac{1}{s-1}\right)$$

So then

$$\mathcal{L}^{-1}\left(e^{-3s}\left(\frac{-1}{s+3} + \frac{1}{s-1}\right)\right) =$$

Table line 2 plus shift because of Theorem 6.3.1.

$$= u_3(t)\left(-e^{-3(t-3)} + e^{t-3}\right)$$

$$= \boxed{u_3(t)\left(e^{t-3} - e^{-3(t-3)}\right)}$$

6. (10 pts) Find the solution of the initial value problem

$$y'' + y = \delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Apply $\mathcal{L}$ to get

$$\mathcal{L}(y)(s^2+1) = \mathcal{L}(\delta(t-\pi)) = e^{-\pi s}$$

So

$$\mathcal{L}(y) = \frac{e^{-\pi s}}{s^2+1}$$

and

$$y = \mathcal{L}^{-1}\left(e^{-\pi s} \frac{1}{s^2+1}\right)$$

$$= \boxed{u_\pi(t) \sin(t-\pi)}$$

Elementary Laplace Transforms

$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s>0$
e^{at}	$\frac{1}{s-a}, \quad s>a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}, \quad s>0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s>0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s>0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s>a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F(\frac{s}{c}), \quad c>0$
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$