Summary of integration techniques

7.1 Integration by parts

1. $\int u dv = uv - \int v du$

2. Integrals of the form $\int p(x)\sin(ax)dx$, $\int p(x)\cos(ax)dx$, $\int p(x)e^{ax}dx$ where $p(x)$ is a polynomial and $a = \text{const}$

Let $u = p(x)$ and, respectively, $dv = \sin(ax)dx$, $dv = \cos(ax)$, and $dv = e^{ax}$; then, respectively, $v = -\frac{1}{a}\cos(ax)$, $v = \frac{1}{a}\sin(ax)$, and $v = \frac{1}{a}e^{ax}$. One integration by parts reduces the degree of $p(x)$ by one, so the number of integration by parts required to complete the problem equals the degree of $p(x)$ (notice that v will be the same in all steps).

3. Integrals $\int p(x)\ln(x)dx$ where $p(x)$ is a polynomial.

Let $u = \ln(x)$, $dv = p(x)dx$ so that $v = \int p(x)dx =$ polynomial in x. Requires only one integration by parts.

4. Integrals $\int (\ln(x))^n dx$

Let $u = (\ln(x))^n$, $dv = dx$, so that $v = x$. After integrating by parts once, you will get $\int (\ln(x))^{n-1}dx$, so the problem requires n integration by parts.

5. Integrals $\int e^{ax}\sin(bx)dx$ and $\int e^{ax}\cos(bx)dx$ where $a, b = \text{const}$

To integrate $\int e^{ax}\sin(bx)dx$, let $u = e^{ax}$, $dv = \sin(bx)dx$ so that $v = -\frac{1}{b}\cos(bx)$, and apply integation by parts. You will get an expression containing $\int e^{ax}\cos(bx)dx$. Then use integration by parts one more time letting $u = e^{ax}$, $dv = \cos(bx)dx$ so that $v = \frac{1}{b}\sin(bx)$. Eventually, you will get an expression of the form

$$\int e^{ax}\sin(bx)dx = \ldots - (\text{positive constant})\int e^{ax}\sin(bx)dx$$

Think of this expression as an *equation* for the unknown integral $\int e^{ax}\sin(bx)dx$ and solve it. Treat integrals $\int e^{ax}\cos(bx)dx$ similarly.

6. There are many other integrals which can be evaluated using integration by parts.
For example, to integrate $\arcsin(x)dx$, let $u = \arcsin(x)$, $dv = dx$ so that $v = x$. Then

$$\int \arcsin(x)dx = x\arcsin(x) - \int x d(\arcsin(x)) = x\arcsin(x) - \int \frac{xdx}{\sqrt{1-x^2}} = x\arcsin(x) + \sqrt{1-x^2} + C$$

(to evaluate $\int \frac{xdx}{\sqrt{1-x^2}}$, one uses the substitution $u = 1 - x^2$).

7.2 Trigonometric Integrals

1. Integrals $\int \sin^m(x)\cos^n(x)dx$

Case 1. At least one of m or n is odd.
If $m = 2k+1$, split off one $\sin(x)$ and let $u = \cos(x)$:

$$\int \sin^m(x)\cos^n(x)dx = \int (\sin^2(x))^k \cos^n(x)(\sin(x)dx) = \int (1-u^2)^k u^n(-du)$$

If $n = 2l+1$, split off one $\cos(x)$ and let $u = \sin(x)$:

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^m(x)(\cos^2(x))^l(\cos(x)dx) = \int u^m(1-u^2)^l du$$

Case 2. Both m and n are even, i.e. $m = 2k$ and $n = 2l$
Use the half-angle identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Then

$$\sin^m(x)\cos^n(x) = \left(\frac{1-\cos(2x)}{2}\right)^k \left(\frac{1+\cos(2x)}{2}\right)^l$$

is a polynomial in $\cos(2x)$, i.e. a sum of powers of $\cos(2x)$. One integrates odd powers as in Case 1 and even powers by reducing the exponent further using identities like $\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$, etc.

2. Integrals $\int \tan^m(x)\sec^n(x)dx$

Case 1. n is even, i.e. $n = 2k$. Split off $\sec^2(x)$ and let $u = \tan(x)$. Then

$$\int \tan^m(x)\sec^n(x)dx = \int \tan^m(x)(\sec^2(x))^{k-1}(\sec^2(x)dx) = \int u^m(u^2+1)^{k-1}du,$$

using the identity $\tan^2(x)+1 = \sec^2(x)$.
Case 2. m is odd, i.e. $m = 2k+1$. Split off $\tan(x)\sec(x)$ and let $u = \sec(x)$. Then

$$\int \tan^m(x)\sec^n(x)dx = \int (\tan^2(x))^k \sec^{n-1}(x)(\tan(x)\sec(x)dx) = \int (u^2-1)^k u^{n-1}du,$$

using the identity $\tan^2(x) = \sec^2(x) - 1$.
Integrals $\int \cot^m(x)\csc^n(x)dx$ are treated similarly using $\cot^2(x)+1 = \csc^2(x)$, $d(\cot(x)) = -\csc^2(x)$ and $d(\csc(x)) = -\cot(x)\csc(x)$.

3. Some special integrals.

(a) $\int \tan(x)dx = \int \dfrac{\sin(x)}{\cos(x)}dx = \int -\dfrac{d(\cos(x))}{\cos(x)} = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$

(b) $\int \sec(x)dx = \int \sec(x)\dfrac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \int \dfrac{\sec^2(x)+\tan(x)\sec(x)}{\sec(x)+\tan(x)}dx = \int \dfrac{d(\sec(x)+\tan(x))}{\sec(x)+\tan(x)} =$
$= \ln|\sec(x)+\tan(x)| + C$

(c) $\int \tan^2(x)dx = \int (\sec^2(x)-1)dx = \tan(x) - x + C$

4. Integrals $\int \sin(mx)\cos(nx)dx$, $\int \sin(mx)\sin(nx)dx$, $\int \cos(mx)\cos(nx)dx$

Use the corresponding identity:

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

7.3 Trigonometric Substitution

1. Integrals containing $\sqrt{a^2-x^2}$. Let $\theta = \arcsin\left(\frac{x}{a}\right)$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $x = a\sin(\theta)$, $dx = a\cos(\theta)$, and

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2(\theta)} = \sqrt{a^2(1-\sin^2(\theta))} = \sqrt{a^2\cos^2(\theta)} = a\cos(\theta),$$

implying that $\cos(\theta) = \frac{\sqrt{a^2-x^2}}{a}$. This (inverse) substitution converts the given integral into an integral involving $\sin(\theta)$ and $\cos(\theta)$. After integrating with respect to θ using methods of 7.2, one converts the resulting function of θ into a function of x.

2. Integrals containing $\sqrt{x^2+a^2}$. Let $\theta = \arctan\left(\frac{x}{a}\right)$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $x = a\tan(\theta)$, $dx = a\sec(\theta)$, and

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2(\theta) + a^2} = \sqrt{a^2(\tan^2(\theta)+1)} = \sqrt{a^2\sec^2(\theta)} = a\sec(\theta),$$

implying that $\sec(\theta) = \frac{\sqrt{x^2+a^2}}{a}$. This (inverse) substitution converts the given integral into an integral involving $\tan(\theta)$ and $\sec(\theta)$. After integrating with respect to θ using methods of 7.2, one converts the resulting function of θ into a function of x.

3. Integrals containing $\sqrt{x^2-a^2}$. Let $\theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$, where $\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$ (notice that this is different from the usual range for $\operatorname{arcsec}(x)$, which is $[0,\pi]$, excluding $\frac{\pi}{2}$; this change is necessary to ensure that $\tan(\theta) \geqslant 0$). Then $x = a\sec(\theta)$, $dx = a\tan(\theta)\sec(\theta)$, and

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2(\theta) - a^2} = \sqrt{a^2(\sec^2(\theta)-1)} = \sqrt{a^2\tan^2(\theta)} = a\tan(\theta),$$

implying that $\tan(\theta) = \frac{\sqrt{x^2-a^2}}{a}$. This (inverse) substitution converts the given integral into an integral involving $\tan(\theta)$ and $\sec(\theta)$. After integrating with respect to θ using methods of 7.2, one converts the resulting function of θ into a function of x.

7.4 Integration of Rational Functions by Partial Fractions

A rational function $\frac{P(x)}{Q(x)}$ is called *proper* if $\deg P < \deg Q$ (for example, $\frac{x+1}{x^3+7x+1}$), and *improper* if $\deg P \geqslant \deg Q$ (for example, $\frac{2x^2}{x^2+5}$). Using long division, any improper fraction can be written as a sum of a polynomial and a proper fraction (cf. Example 1 on p. 490), which reduces the problem of evaluating $\int \frac{P(x)}{Q(x)}dx$ to the case where the fraction $\frac{P(x)}{Q(x)}$ is proper. Integration of these is based on the fact that any proper fraction can be written as a sum of *partial* fractions. A partial fraction is a fraction of one of the following two types: $\frac{A}{(ax+b)^n}$, or

$\dfrac{Ax+B}{(ax^2+bx+c)^n}$ where $b^2-4ac<0$ and $A, B = \text{const}$. To find a partial fraction decomposition of a given proper fraction $\dfrac{P(x)}{Q(x)}$, one first needs to factor the denominator:

$$Q(x) = (p_1x+q_1)^{m_1}\cdots(p_sx+q_s)^{m_s}(a_1x^2+b_1x+c_1)^{n_1}\cdots(a_tx^2+b_tx+c_t)^{n_t}.$$

The general rule is that the factor $(px+q)^m$ contributes the following fragment to the decomposition of $\dfrac{P(x)}{Q(x)}$:

$$\frac{P(x)}{Q(x)} = \cdots + \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m} + \cdots,$$

the factor $(ax^2+bx+c)^n$ contributes

$$\frac{P(x)}{Q(x)} = \cdots + \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n} + \cdots,$$

and the full decomposition is the union of these fragments written for each factor in the factorization of $Q(x)$. One first writes a general decomposition of $\dfrac{P(x)}{Q(x)}$ with undetermined coefficients, and then determines those coefficients from a system of linear equations obtained by equating coefficients of equal powers of x in the left- and right-hand sides. For example, if Q is a product of distinct linear factors, $Q(x) = (x-\alpha_1)\cdots(x-\alpha_s)$ then the decomposition of any proper fraction with denominator Q will be of the form:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_s}{x-\alpha_s}$$

Another example: if $Q(x) = (x+1)^3(x^2+5)^2$, then

$$\frac{P(x)}{Q(x)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+5} + \frac{Fx+G}{(x^2+5)^2}$$

After a decomposition of $\dfrac{P(x)}{Q(x)}$ into a sum of partial fractions have been found, to integrate $\displaystyle\int \frac{P(x)}{Q(x)}$, it remains to integrate each partial fraction. Clearly,

$$\int \frac{dx}{(x-\alpha)^n} = \frac{(x-\alpha)^{-n+1}}{-n+1} + C \quad \text{if} \quad n \neq -1, \quad \text{and} \quad \ln|x-\alpha| + C \quad \text{otherwise.}$$

To integrate $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}dx$, where $b^2-4ac<0$, first complete the square in the denominator; this will reduce your problem to an integral of the form $\displaystyle\int \frac{Cu+D}{u^2+d^2}du$, which is easily evaluated using

$$\int \frac{udu}{u^2+d^2} = \frac{1}{2}\ln(u^2+d^2) + C \quad \text{and} \quad \int \frac{du}{u^2+d^2} = \frac{1}{d}\arctan\left(\frac{u}{d}\right) + C.$$

In addition,

$$\int \frac{udu}{(u^2+d^2)^n} = \frac{1}{2}\frac{(u^2+d^2)^{-n+1}}{-n+1} + C \quad \text{for} \quad n \neq -1.$$

Some integrals can be reduced to integrals of rational functions. For example, if your integral contains $\sqrt[n]{g(x)}$, where $g(x)$ is a polynomial, try to substitute $u = \sqrt[n]{g(x)}$ (this substitution always works if $g(x)$ is linear). Next, you can always integrate $\int R(\sin(x), \cos(x))dx$, where $R(\sin(x), \cos(x))$ is a rational function of $\sin(x)$ and $\cos(x)$ by letting $t = \tan\left(\dfrac{x}{2}\right)$ (cf. Problem 57). Then

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad dx = \frac{2}{1+t^2}dt$$

(because $x = 2\arctan(t)$). Thus, the given integral gets reduced to an integral of a rational function of t.