Math 223, Spring 2009
Midterm 1 Preparation

Your first midterm is on Thursday, March 12. It is a closed-note, closed-book, open-brain, no-calculator 70-minute examination held during the regular class time. It will cover the material up to and including Chapter 9.

The exam will contain 6–7 questions, some with parts. You may be asked to recall some definitions or reproduce statements of theorems or proofs listed on this sheet. You might also see some problems that have already appeared in class, on homework assignments, or quizzes. Another portion of the exam will consist of problems you had not see before, i.e. you will have to solve new problems using techniques you already know. This portion will not comprise the bulk of the exam, and most (but not all) of the exam will have the format and difficulty level of an average homework problem.

A review session will take place on Wednesday, March 11, 7–8 pm, in SCI 364. Please come prepared with questions. I will not prepare any materials for the review session beyond this document and will only answer your question during the hour.

What you need to commit to memory

(1) You must know the definitions of the following.

(a) Odd and even numbers
(b) Prime number
(c) Divisibility of an integer by another
(d) Greatest common divisor
(e) Least common multiple
(f) Congruence
(g) Least residue of a number modulo another

(2) You must know the statements of the following.

(a) Pythagorian Triples Theorem
(b) Linear Equation Theorem
(c) Fundamental Theorem of Arithmetic
(d) Prime Divisibility Property
(e) Linear Congruence Theorem
(f) Fermat's Little Theorem

Topics you must study

(1) Propositional logic, truth tables, basic logic identities (deMorgan laws etc.)
(2) Forming converse, contrapositive, and inverse of an implication
(3) Propositional functions and quantifies
(4) Recognizing direct proofs, proofs by induction, contradiction, contrapositive, and existence and uniqueness proofs
(5) Divisibility
(6) Prime factorization
(7) Congruences
(8) Fermat's Little Theorem

Proofs you must learn

(1) Proof that $\sqrt{2}$ is irrational
(2) Proof of Prime Divisibility Property
(3) Proof of the Fundamental Theorem of Arithmetic
(4) Proof of Linear Congruence Theorem

(5) Proof of the lemma stating that one can cancel a common factor in a congruence with an assumption on some *gcd* (this is one of the lemmas used in the proof of Fermat's Little Theorem)
(6) Proof of the lemma stating that multiples of all the integers up to some prime are the same as those integers modulo that prime, possibly in a different order and again with an assumption on some *gcd* (this is another lemma used in the proof of Fermat's Little Theorem)
(7) Proof of Fermat's Little Theorem

Computational problems you must be able to do

(1) Computing the greatest common divisor using Euclidean Algorithm
(2) Solving $ax + by = gcd(a, b)$ using Linear Equation Theorem
(3) Finding the number of divisors of an integer from its prime factorization
(4) Solving linear congruences using algebra, inspection, or Linear Congruence Theorem
(5) Applying Fermat's Little Theorem to showing a number is not prime, finding least residues of congruences of the sort $a^b \pmod p$, and finding solutions to congruence equations of the form $x^a \equiv c \pmod p$

How to study for the exam

The exam will be friendly to those who have studied carefully and followed all the instructions on this sheet. Most of the test questions will look familiar. As mentioned above, you will be asked to repeat some definitions, state some theorems, and reproduce some proofs you have seen before. The exam will contain some exercises you have not seen before, but they will not comprise the bulk of the exam, and there will be no questions that only divine intervention will help you solve. You will do poorly if you fail to follow the advice on this preparation sheet.

(1) Read this worksheet thoroughly.
(2) Read and understand your class notes.
(3) Know how to do all the homework and quiz problems. The solutions are on our class conference.
(4) Go to office hours to ask questions.
(5) After you have done all of the above, start on the review questions below.

Review Problems (solutions will be provided later)

(1) Prove by induction.
   (a) For all $n \in \mathbb{N}$, we have $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.
   (b) For all $n \in \mathbb{N}$, $n \geq 4$, we have $2^n < n!$.

(2) Let $a, b \in \mathbb{Z}$ and suppose $gcd(a, b) = 1$. Prove the following.
   (a) $gcd(a + b, a - b) = 1$ or 2.
   (b) $gcd(a + 2b, 2a + b) = 1$ or 3.
   (c) $gcd(a^n, b^n) = 1$ for all $n \in \mathbb{N}$.

(3) Let $a, b, c \in \mathbb{Z}$ and let $g = gcd(a, c)$. Prove that if $c|ab$, then $c|gb$.

(4) Decide if the following are true or false. If true, provide a proof. If false, provide a counterexample.
   (a) For all $k, n, r \in \mathbb{Z}$, $gcd(k, n) = gcd(k, n + rk)$.
   (b) For all $a, b, n \in \mathbb{Z}$, if $a^2|n$, $b^2|n$, and $a^2 < leqb^2$, then $a|b$.
   (c) For all $a, b, n \in \mathbb{N}$, if $a^n|b^n$, then $a|b$.

(5) Kim Bottomly throws a party and orders apples and oranges at a total cost of \$8.39. If apples cost her 25 cents and oranges 18 cents each, how many of each type did she order?

(6) Kim Bottomly believes that she has 50 coins, all of which are pennies, dimes and quarters, with a total worth of 3 dollars. Determine whether her computations are possible.

(7) Prove that, if a is an odd integer, then $a^2 \equiv 1 \pmod 8$. (Hint: All integers are of the form $4k + r$ where $k, r \in \mathbb{Z}$ and $0 \leq r < 4$. How about *odd* integers?)

(8) For $n \in \mathbb{N}$, suppose $n \equiv 3 \pmod 4$. Prove that n cannot be a sum of squares of two integers. (Hint: Use proof by contradiction.)

(9) Determine all values of x mod 301 such that $140x \equiv 133 \pmod{301}$. You will need to know that $gcd(140, 301) = 7$.

(10) Use Fermat's Little Theorem to do the following.
   (a) Find the least residue of 1945^{12} modulo 11.
   (b) Solve $x^{212} \equiv 6 \pmod 7$.