Math 225, Fall 2008
Midterm 1 Preparation

Your first midterm is on Thursday, October 23. It is a closed-note, closed-book, open-brain, no-calculator 70-minute examination held during the regular class time. It will cover the material up to and including the October 6 lecture. Therefore any material up to and including Burnside’s Lemma is fair game for the exam. Remember that my handout 225CountingSummary.pdf might also be useful to you as you study for the exam.

The exam will contain 6–7 questions, some with parts. You may be asked to recall some definitions or reproduce statements of theorems or proofs listed on this sheet. You might also see some problems that have already appeared in class, on homework assignments, or quizzes. Another portion of the exam will consist of problems you had not see before, i.e. you will have to solve new problems using techniques you already know. This portion will not comprise the bulk of the exam, and most of it will have the format and difficulty level of an average homework problem.

A review session will take place on Wednesday, October 22, in class. Please come prepared with questions. Chelsea will also hold a review session Wednesday evening during her regular office hours.

What you need to commit to memory

(1) You must know the definitions of the following.

(a) Negation, conjunction, disjunction, implication, and equivalence (defined through truth tables)
(b) Converse, contrapositive, inverse
(c) Function, injection, surjection, bijection
(d) Cardinality of a set
(e) What it means for two sets to have the same cardinality
(f) Countable set
(g) Power set of a set
(h) Permutation and r-permutation
(i) r-combination
(j) Group, abelian group
(k) Order of a group
(l) Permutation group, cyclic group, and dihedral group
(m) Invariant set

(2) You must know the statements and proofs (unless otherwise indicated) of the following.

(a) Fundamental Theorem of Arithmetic
(b) The proof that the set of positive rational numbers is countable
(c) Multiplication Principle, Addition Principle, and Inclusion-Exclusion for two sets (these are just statements)
(d) Proofs of theorems giving the number of
   (i) r-permutations
   (ii) r-combinations
   (iii) r-permutations with repetitions
   (iv) permutations with indistinguishable objects
   (v) r-combinations with repetition
(e) Binomial Theorem
(f) Pascal’s Identity (statement only)
(g) Burnside’s Lemma (statement only)
Topics you must study

1. Propositional logic, truth tables
2. Types of proofs, especially induction
3. Cardinality and countability
4. Multiplication, Addition, and Inclusion-Exclusion Principles
5. Arrangements and selections, possibly with repetition
6. Different interpretations of arrangements and selections with repetition (see handout on counting for a concise explanation)
7. Working with and manipulating the binomial formula
8. Counting in the presence of symmetry
9. Basic group theory and symmetry groups
10. Interpretations of $C_n$ and $D_n$ as symmetry groups
11. Using Burnside’s Lemma

Computational problems you must be able to do

1. Manipulating logical expressions and going back and forth between English and logic
2. Move quantifiers through expressions
3. Verifying a set is countable
4. Using Multiplication, Addition, and Inclusion-Exclusion Principles in various situations
5. Counting using permutations (arrangements) and combinations (selections) with or without repetition
6. Evaluating the binomial formula and finding the sum of a polynomial by manipulating it
7. Counting colorings while keeping track of symmetry
8. Verifying a set is a group
9. Moving back and forth between the definition of $S_n$ as a set of bijective maps and interpretation of such maps by two-row matrices
10. Manipulating permutations in two-row matrix forms (i.e finding inverses and compositions of permutations in this form)
11. Describing elements of a symmetry group of an object
12. Finding the number of distinct colorings using Burnside’s Lemma

How to study for the exam

The exam will be friendly to those who have studied carefully and followed all the instructions on this sheet. As mentioned above, many test questions will look familiar. You will be asked to repeat some definitions, state some theorems, and reproduce some proofs you have seen before. The exam will contain some exercises you have not seen before, but they will not comprise the bulk of the exam.

1. Read this worksheet thoroughly.
2. Read and understand your class notes.
3. Know how to do all the homework and quiz problems. The solutions are on our class conference.
4. Go to office hours to ask questions.
5. After you have done all of the above, start on the review problems below.
Review Problems (solutions will be provided later)

(1) Construct the truth table for the expression \( p \implies (\neg q \land r) \).

(2) Let \( B(x, y) \) be the statement “\( y \) is the best friend of \( x \)”. Write the statement “everyone has exactly one best friend” as a logical expression using \( B(x, y) \).

(3) Use induction to prove that \( n^3 - n \) is divisible by 3 for all \( n \geq 1 \).

(4) Give an example of a function \( f : \mathbb{N} \to \mathbb{N} \) that is
   (a) Injective but not surjective
   (b) Surjective but not injective
   (c) Both injective and surjective
   (d) Neither injective nor surjective

(5) How many ways are there to select a committee to develop a discrete math course at a school if the committee is to consist of three faculty members from the math department and four from the computer science department and they have 9 and 11 faculty members, respectively?

(6) A group contains \( n \) men and \( n \) women. How many ways are there to arrange these people so that men and women alternate?

(7) How many ways are there to select five bills from a cash box containing $1, $2, $5, $10, $20, $50, and $100 bills.

(8) How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

(9) How many ways are there to travel in four-dimensional \( xyzw \)-space from the origin \((0,0,0,0)\) to the point \((4,3,5,4)\) by taking steps on unit in the positive \( x \), positive \( y \), positive \( z \), or positive \( z \) direction?

(10) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

(11) What is the coefficient of \( x^{12}y^{13} \) in the binomial expansion of \((x + y)^{25}\)?

(12) Section 5.5, problem 14(f)

(13) Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \). Find \( \beta \circ \alpha, \alpha \circ \beta, \alpha^{-1}, \text{ and } (\alpha \circ \beta)^{-1} \).

(14) Section 9.1, problem 3(c)

(15) Section 9.1, problem 25(a)

   Note that there are no problems here on Burnside’s Lemma. This is because you are turning in the homework set on this topic a few days before the exam, so those homework problems can be also be thought of as review. If you’d like more practice problems on Burnside’s Lemma, please see me.