Math 305, Fall 2007
Final Exam

Name: ______________________________ Student ID: ___________________________

Directions: Check that your test has 13 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 2.5 hours. Good luck!

1. (20 points) _____________
2. (10 points) _____________
3. (8 points) _____________
4. (5 points) _____________
5. (8 points) _____________
6. (5 points) _____________
7. (5 points) _____________
8. (10 points) _____________
9. (7 points) _____________
10. (5 points) _____________
11. (7 points) _____________

Total (out of 90): _____________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

   (a) Give an example of an infinite nonabelian group.

   (b) Give an example of an infinite group all of whose elements have finite order.

   (c) Give representatives of all the isomorphism classes of groups of order 12.

   (d) Compute the number of elements in $\mathbb{Z}_{100}/\langle [6], [8] \rangle$.

   (e) Give an example of a ring which is not unital.

   (f) Give an example of a commutative unital ring which is not an integral domain.

   (g) Give an example of a ring homomorphism from $\mathbb{Z}_{10}$ to $\mathbb{Z}_5$.

   (h) Find the characteristic of $\mathbb{Z}_3[x] \times \mathbb{Z}_3$.

   (i) Give a polynomial $f(x) \in \mathbb{Q}[x]$ such that $\mathbb{Q}[x]/(f(x))$ is not a field.

   (j) Give an example of a principal ideal domain.

2. (2 pts each) State the following theorems.

   (a) Lagrange's Theorem

   (b) Fundamental Theorem of Arithmeic

   (c) Cayley's Theorem

   (d) Fundamental Theorem of Algebra

   (e) Division Algorithm

3. (4 pts each)

(a) Determine whether $(G, \cdot)$, where $G = \{2^m 3^n : m, n \in \mathbb{Z}\}$ and $\cdot$ is the usual multiplication on real numbers, is a group.

(b) Determine whether the set $\{\sigma \in S_4 \colon \sigma(2) \neq 3\}$ is a subgroup of S_4.

4. (5 pts) Show that, if H is a subgroup of $\mathbb{Z}$, then it has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

5. (4 pts each)

(a) Show that $\langle (1\,2\,3) \rangle$ is a normal subgroup of S_3.

(b) List the elements of $S_3 / \langle (1\,2\,3) \rangle$ and identify this quotient as a more familiar group.

6. (5 pts) Use extended version of Sylow's Theorem to show that a group of order 175 must have normal subgroups of orders 7 and 25.

7. (5 pts) Consider the four rings $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}_2[x]$, and $\mathbb{Q}$. Prove that each of these is not ring isomorphic to the other two.

8. (5 pts each) Suppose $\theta\colon R \to S$ is a ring isomorphism.

(a) Show that, if R is an integral domain, so is S.

(b) Show that the characteristics of R and S are equal.

9. (7 pts) The *center* of a ring R is defined to be $Z(R) = \{c \in R : cr = rc \text{ for every } r \in R\}$. Suppose that $\phi \colon R \to S$ is a surjective ring homomorphism. Prove that the image of the center of R is contained in the center of S.

10. (5 pts) Prove that a ring F is a field if an only if the only ideals of F are $\{0\}$ and F itself.

11. (7 pts) State and prove the First Isomorphism Theorem for rings. Do not assume the First Isomorphism Theorem for groups.