Math 305, Fall 2007
Final Exam Solutions

Name: ______________________________ Student ID: ___________________________

Directions: Check that your test has 13 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 2.5 hours. Good luck!

1. (20 points) ______________
2. (10 points) ______________
3. (8 points) ______________
4. (5 points) ______________
5. (8 points) ______________
6. (5 points) ______________
7. (5 points) ______________
8. (10 points) ______________
9. (7 points) ______________
10. (5 points) ______________
11. (7 points) ______________

Total (out of 90): ______________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

   (a) Give an example of an infinite nonabelian group.

   Solution: $SL(n, \mathbb{R})$ or $\mathbb{Z} \times S_3$ (there are many others, of course).

   (b) Give an example of an infinite group all of whose elements have finite order.

   Solution: $\mathbb{Q}/\mathbb{Z}$

   (c) Give representatives of all the isomorphism classes of groups of order 12.

   Solution: $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$.

   (d) Compute the number of elements in $\mathbb{Z}_{100}/\langle [6], [8] \rangle$.

   Solution: Since $|\langle [6], [8] \rangle| = |\langle [2] \rangle| = 50$, $|\mathbb{Z}_{100}/\langle [6], [8] \rangle| = 100/50 = 2$.

   (e) Give an example of a ring which is not unital.

   Solution: $n\mathbb{Z}$ for any $n \neq 1$.

   (f) Give an example of a commutative unital ring which is not an integral domain.

   Solution: $\mathbb{Z}_n$ for any n composite.

   (g) Give an example of a ring homomorphism from $\mathbb{Z}_{10}$ to $\mathbb{Z}_5$.

   Solution: $\theta \colon \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $\theta([k]_{10}) = [k]_5$.

   (h) Find the characteristic of $\mathbb{Z}_3[x] \times \mathbb{Z}_3$.

   Solution: 3

   (i) Give a polynomial $f(x) \in \mathbb{Q}[x]$ such that $\mathbb{Q}[x]/(f(x))$ is not a field.

   Solution: Any reducible polynomial would do, such as $f(x) = x^2 - 1$.

   (j) Give an example of a principal ideal domain.

   Solution: $\mathbb{Z}$ or $F[x]$ where F is a field.

2. (2 pts each) State the following theorems.

   (a) Lagrange's Theorem

   Solution: See class notes from October 15.

   (b) Fundamental Theorem of Arithmetic

   Solution: See class notes from September 27.

   (c) Cayley's Theorem

   Solution: See class notes from October 22.

   (d) Fundamental Theorem of Algebra

   Solution: See class notes from November 25.

   (e) Division Algorithm

   Solution: See class notes from November 28.

3. (4 pts each)

(a) Determine whether $(G, \cdot)$, where $G = \{2^m 3^n : m, n \in \mathbb{Z}\}$ and $\cdot$ is the usual multiplication on real numbers, is a group.

Solution: This was problem 4(a) on review problems for the first midterm.

(b) Determine whether the set $\{\sigma \in S_4 \colon \sigma(2) \neq 3\}$ is a subgroup of S_4.

Solution: This was problem 7 on review problems for the first midterm.

4. (5 pts) Show that, if H is a subgroup of $\mathbb{Z}$, then it has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

Solution: This was problem 6 on first midterm.

5. (4 pts each)

   (a) Show that $\langle(1\,2\,3)\rangle$ is a normal subgroup of S_3.

   Solution: This was problem 6(a) on second midterm.

   (b) List the elements of $S_3/\langle(1\,2\,3)\rangle$ and identify this quotient as a more familiar group.

   Solution: This was problem 6(b) on second midterm.

6. (5 pts) Use extended version of Sylow's Theorem to show that a group of order 175 must have normal subgroups of orders 7 and 25.

Solution: This was problem 55.13 on homework 9.

7. (5 pts) Consider the three rings $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}_2[x]$, and $\mathbb{Q}$. Prove that each of these is not ring isomorphic to the other two.

Solution: Since $\mathbb{Q}$ is a field, and neither $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_2[x]$ is, $\mathbb{Q}$ is not isomorphic to either. The characteristic of $\mathbb{Z} \times \mathbb{Z}$ is 0 but characteristic of $\mathbb{Z}_2[x]$ is 2, so they are not isomorphic.

8. (5 pts each) Suppose $\theta\colon R \to S$ is a ring isomorphism.

   (a) Show that, if R is an integral domain, so is S.

   Solution: This was problem 27.3 on homework 11.

   (b) Show that the characteristics of R and S are equal.

   Solution: This was problem 27.15 on homework 11.

9. (7 pts) The *center* of a ring R is defined to be $Z(R) = \{c \in R : cr = rc \text{ for every } r \in R\}$. Suppose that $\phi \colon R \to S$ is a surjective ring homomorphism. Prove that the image of the center of R is contained in the center of S.

Solution: Suppose $s \in S$ is in the image of $Z(R)$, i.e. $s = \phi(r)$ for some $r \in Z(R)$. We want to show that s is an element of $Z(S)$, i.e. $ds = sd$ for all $d \in S$. Given $d \in S$, since ϕ is surjective, $d = \phi(a)$ for some $a \in R$ and we have

$$ds = \phi(a)\phi(r) = \phi(ar) = \phi(ra) = \phi(r)\phi(a) = sd.$$

The middle equality is true since r is in $Z(R)$ and equalities before and after the middle one are true since ϕ is a homomorphism.

10. (5 pts) Prove that a ring F is a field if an only if the only ideals of F are $\{0\}$ and F itself.

Solution: See class notes from 12/5.

11. (7 pts) State and prove the First Isomorphism Theorem for rings. Do not assume the First Isomorphism Theorem for groups.

Solution: See class notes from 10/29 for the group version and 12/5 for generalization to rings.