Math 305, Fall 2007
Final Exam Preparation

Your final exam is self-scheduled and can be taken any time during the finals period. It is a closed-note, closed-book, open-brain, no-calculator 2.5-hour examination. The exam is cumulative, and will have the same format as the midterms except it will be roughly twice as long. Later material will be emphasized more heavily.

What follows are some guidelines for studying the portion of the material covered since the second midterm. To study for earlier topics, use review handouts for first and second midterms.

What you need to commit to memory

(1) You must know the definitions of the following.

- (a) Ring, commutative ring, unit, subring
- (b) Polynomial ring
- (c) Zero divisor, integral domain
- (d) Field, subfield
- (e) Ring homomorphism and isomorphism
- (f) Characteristic
- (g) Embedding of a ring in another
- (h) Extension of a ring
- (i) Algebraic element, algebraic extension, algebraic closure, algebraically closed field
- (j) Irreducible element
- (k) Unique factorization domain
- (l) Euclidean domain
- (m) Ideal, principal ideal, principal ideal domain
- (n) Quotient ring

(2) You must know the statements and proofs of the following.

- (a) Extended version of Sylow's Theorem (statement only)
- (b) Statement that if R is an integral domain then it has characteristic 0 or prime p
- (c) Statement that if an integral domain has characteristic 0 then $\mathbb{Z}$ embeds in it
- (d) Fundamental Theorem of Algebra (statement only)
- (e) Division Algorithm (statement only)
- (f) Remainder Theorem
- (g) Factor Theorem (statement only)
- (h) Statement that if F is a field, then $F[x]$ is a UFD
- (i) Statement that a Euclidean domain is a UFD (statement only)
- (j) Statement that F is a field if and only if the only ideals of F are $\{0\}$ and F itself
- (k) Statement that $\mathbb{Z}$ is a PID
- (l) Statement that a kernel of a ring homomorphism is an ideal
- (m) First Isomorphism Theorem for rings
- (n) Statement that $F[x]/p(x)$ is a field iff $p(x)$ is irreducible

Topics you must study

(1) Properties of rings and fields
(2) Ring homomorphisms and isomorphisms
(3) Characteristic
(4) Algebraic extensions and closures
(5) Polynomial rings
(6) Irreducibility and unique factorization domains
(7) Euclidean domains
(8) Ideals
(9) Principal ideals and PIDs
(10) Quotient rings
(11) First Isomorphisms Theorem for rings
(12) Relationships between various kinds of domains

Computational problems you must be able to do

(1) Verifying that a set is a (sub)ring or a (sub)field

(2) Showing something is or is not an integral domain
(3) Verifying a map is or is not a ring homomorphism or isomorphism
(4) Verifying that a ring property is preserved by an isomorphism
(5) Finding the characteristic of a ring
(6) Deciding if a field is algebraically closed
(7) Deciding if a ring is a UFD or a Euclidean domain
(8) Verifying a subring is a (principal) ideal
(9) Forming quotient rings

How to study for the exam

The exam will be friendly to those who have studied carefully and followed all the instructions on this sheet. Most of the test questions will look familiar. You will be asked to repeat some definitions, state some theorems, and reproduce some proofs you have seen before. The exam will contain some exercises you have not seen before, but they will not comprise the bulk of the exam, and there will be no questions that only divine intervention will help you solve. You will do poorly if you fail to follow the advice on this preparation sheet.

(1) Read this worksheet thoroughly.
(2) Read and understand your class notes.
(3) Know how to do all the homework and quiz problems. The solutions are on our class conference.
(4) Go to office hours to ask questions.
(5) After you have done all of the above, start on the review problems below.

Review Problems (solutions will be provided later)

(1) For each of the following sets, determine if it is a ring. If so, decide if it is an integral domain. Supply a proof only for a *negative* answer.
  (a) The set of diagonal $n \times n$ matrices for a fixed $n \in \mathbb{Z}_{\geq 1}$. Note: we say that a square matrix $A = (a_{ij})$ is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$.
  (b) The set of all $f \in M(\mathbb{R})$ such that $f(q) = 0$ for all $q \in \mathbb{Q}$.
  (c) The set of all $f \in \mathrm{Cont}([0,1], \mathbb{R})$ for which $\int_0^1 f(x)\,dx = 0$. Here $\mathrm{Cont}([0,1], \mathbb{R})$ means the set of continuous functions $f\colon [0,1] \to \mathbb{R}$.
  (d) The set of rational numbers $\frac{m}{n}$ such that 3 does not divide n when $\frac{m}{n}$ is written in lowest terms. Note: obviously $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$.

(2) (a) Prove that a ring R is commutative iff $a^2 - b^2 = (a+b)(a-b)$ for all $a, b \in R$.
  (b) Let R be a ring such that $x^2 = x$ for all $x \in R$. Prove that R is commutative. Such a ring is called a *Boolean ring*. (Hint: Consider both the quantities $(a+b)^2$ and $(ab)^2$.)

(3) (a) Prove that, if $\phi\colon R \to S$ is a ring isomorphism, then $\phi^{-1}\colon S \to R$ is also a ring isomorphism.
  (b) Let C be the collection of all rings. For all R and S in C, we say that $R \sim S$ if there is a ring isomorphism $\phi\colon R \to S$. Prove that $\sim$ is an equivalence relation on C.

(4) The following are subrings of the ring $\mathbb{Z}[x]$. Decide whether or not each is an ideal, giving a proof only if it is *not* an ideal.
  (a) The subring I of all polynomials whose constant term is a multiple of 3.
  (b) The subring $\mathbb{Z}[x^2]$ of all polynomials with even power terms.
  (c) The subring I of polynomials p such that $p'(0) = 1$. Here $p'(0)$ is the first derivative of p evaluated at 0.
  (d) The subring I of polynomials p such that $p(0) = p'(0) = 0$.

(e) The subring I of polynomials whose coefficients sum to zero. (Hint: write this condition in a different way.)

(5) Let $\phi\colon R \to S$ be a ring homomorphism. Prove that, if ϕ is surjective and I is an ideal of R, then $\phi(I)$ is an ideal of S. Give an example where this statement fails if ϕ is not surjective.

(6) Prove that the following rings are not isomorphic.
  (a) $\mathbb{Z}_5$ and $\mathbb{Z}_8$
  (b) $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$
  (c) $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$
  (d) $\mathbb{Z}$ and $\mathbb{Q}$
  (e) $\mathbb{R}$ and $\mathbb{C}$ (hint: assume that there is an isomorphism $\phi\colon \mathbb{C} \to \mathbb{R}$ and show that $\phi(i)$ cannot be defined)
  (f) $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$

(7) Let R be a commutative ring and let I be an ideal of R. Let $\operatorname{rad} I$ be the subset of R defined by $\operatorname{rad} I = \{\, r \in R\colon\ r^n \in I \text{ for some } n \in \mathbb{Z}_{\geq 1} \,\}$.
  (a) Prove that $\operatorname{rad} I$ is an ideal of R.
  (b) Prove that $(\operatorname{rad} I)/I = N(R/I)$, where $N(R/I)$ is the ideal of nilpotent elements of R/I. (Hint: prove that each set is a subset of the other. This problem looks scary but is not.)

(8) Let x be a nilpotent element of a commutative unital ring R. Recall that a *unit* is an element of a unital ring R with a multiplicative inverse.
  (a) Prove that x is either zero or a zero divisor.
  (b) Prove that rx is nilpotent for all $r \in R$.
  (c) Prove that $1 + x$ is a unit in R.
  (d) Prove that the sum of a nilpotent element and a unit is a unit.

(9) Find q and r guaranteed by the Division Algorithm for the given a and b in the given polynomial ring $F[x]$.
  (a) $a = x^4 + 3x + 2$ and $b = 2x + 1$ in $\mathbb{Q}[x]$
  (b) $a = x^2 + ix$ and $b = ix + 2$ in $\mathbb{C}[x]$
  (c) $a = 3x^3 + 2x + 1$ and $b = 2x + 1$ in $\mathbb{Z}_5[x]$

(10) Decide if each of the following polynomials is reducible in the given polynomial rings. If so, write down its factorization into irreducible elements. No explanation required. (Hint: A reducible quadratic or cubic must have a root. A reducible quartic will either have a root or will factor into two irreducible quadratics. If it factors into two quadratics, then it can be expressed as $(x^2+ax+b)(x^2+cx+d)$ for some $a, b, c, d \in F$. Multiply this expression out, collect terms and you should be able to solve for a, b, c, d.)
  (a) $p = x^2 + x + 1$ in $\mathbb{Z}_2[x]$
  (b) $p = x^2 + 1$ in $\mathbb{Z}_3[x]$
  (c) $p = x^3 + 1$ in $\mathbb{Z}_2[x]$
  (d) $p = x^4 + 1$ in $\mathbb{Z}_5[x]$

(11) In each of the following examples is a homomorphism $\theta\colon F[x] \to E$ for various fields F and E. The function θ is given by $\theta(f) = f(c)$ for some fixed $c \in E$. In each case, identify the kernel and image of θ explicitly and write down the isomorphism guaranteed by the First Isomorphism Theorem for rings.
  (a) $\theta : \mathbb{Q}[x] \to \mathbb{Q}$ given by $\theta(f) = f(3)$
  (b) $\theta : \mathbb{Q}[x] \to \mathbb{R}$ given by $\theta(f) = f(1 + \sqrt{2})$
  (c) $\theta : \mathbb{R}[x] \to \mathbb{C}$ given by $\theta(f) = f(2 - i)$