Math 305, Fall 2007
First Midterm Exam, October 11, 2007

Name: ____________________________ Student ID: _________________________

Directions: Check that your test has 10 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (10 points) _____________
2. (8 points) _____________
3. (5 points) _____________
4. (8 points) _____________
5. (6 points) _____________
6. (5 points) _____________
7. (8 points) _____________
8. (5 points) _____________

Total (out of 55): _____________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

   (a) Give an example of a noncommutative relation $*$ on $\{1, 2\}$.

   (b) Give an example of an infinite abelian group. Be sure to include the operation.

   (c) Give an example of a finite nonabelian group. Be sure to include the operation.

   (d) Give an example of a symmetry group of an object in the plane.

   (e) Give the number of elements of A_n, $n \geq 3$.

2. (4 pts each)

(a) Define precisely what it means for a function $f\colon S \to T$ to be *injective* and *surjective*.

(b) Let $M_2(\mathbb{R})$ denote the set of 2×2 matrices with real coefficients and recall that the determinant of a 2×2 matrix is given by the formula $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. We can thus think of the determinant as a map $\det\colon M_2(\mathbb{R}) \to \mathbb{R}$. Show that this map is surjective but not injective.

3. (5 pts) Verify that the set $G = \{2^m 3^n \colon m, n \in \mathbb{Z}\}$ is a group with respect to multiplication.

4. (4 pts each) Let $(G, *)$ be a group and suppose that A and B are subgroups of G. Define a subset $A * B$ of G by the following:

$$A * B = \{a * b \colon a \in A,\ b \in B\}.$$

(a) Consider $G = S_3$ and let $A = \{(1), (12)\}$ and $B = \{(1), (13)\}$. List the elements of $A * B$ and prove that $A * B$ is not a subgroup of G.

(b) Suppose that $(G, *)$ is abelian. Prove that, for any subgroups A and B of G, the subset $A * B$ is a subgroup of G.

5. (3 pts each)

(a) Show that S_n is nonabelian for $n \geq 3$.

(b) Decide whether $(245)(1354)(125) \in S_5$ is an even or an odd permutation.

6. (5 pts) Show that, if H is a subgroup of $\mathbb{Z}$, then it has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

7. (4 pts each)

(a) For G a group with operation $*$ and for $a \in G$ define the *centralizer* of a in G to be the set

$$C(a) = \{x \in g \colon a * x = x * x\}.$$

Show that $C(a)$ is a subgroup of G.

(b) Find the centralizer of 2 in $\mathbb{Z}$.

8. (5 pts) Let G be a group and define a relation $\sim$ on G by the following: for all $a, b \in G$, $a \sim b$ iff there is $g \in G$ such that $b = gag^{-1}$. Prove that $\sim$ is an equivalence relation on G.