Math 350, Fall 2007
Midterm 1 Preparation

Your first midterm is on Thursday, October 11. It is a closed-note, closed-book, open-brain, no-calculator 70-minute examination held during the regular class time. It will cover the material up to and including Section 13. A review session will take place on Wednesday, October 10, in class.

What you need to commit to memory

(1) You must know the definitions of the following.

(a) Mapping (or map, or function)
(b) Domain, codomain (or range)
(c) Injection (one-to-one), surjection (onto)
(d) Composition
(e) Operation
(f) Group, subgroup
(g) Abelian group
(h) Order of a group
(i) Permutation group, cycle, transposition
(j) Alternating group
(k) Equivalence relation
(l) Partition
(m) Congruence
(n) Greatest common divisor
(o) Euler's Phi Function

(2) You must know the statements and proofs of the following.
(a) Theorem with four parts about when two maps and their composition are injective or surjective
(b) Theorem that a map is invertible iff it is a bijection
(c) Theorem that inverses and identity in a group are unique
(d) Statement that order of S_n is $n!$
(e) Statement that every subgroup of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$
(f) Theorem about the one-to-one correspondence between equivalence relations and partitions of a set
(g) Statement that there are n congruence classes modulo n.
(h) Prime Divisibility Property
(i) Fundamental Theorem of Arithmetic (statement only)

Topics you must study

(1) Forming converse, contrapositive, and inverse of an implication
(2) Propositional functions and quantifies
(3) Recognizing direct proofs, proofs by induction, contradiction, contrapositive, and existence and uniqueness proofs
(4) Maps, compositions of maps
(5) Operations
(6) Examples of groups: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}^\times$, $\mathbb{R}^\times$, $\mathbb{C}^\times$, $\{0\}$, $\{1\}$, $\{1, -1\}$, $n\mathbb{Z}$ ($n \in \mathbb{Z}$), $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{Sym}(\mathbb{R}^2)$, S_n, A_n, D_4, $\mathbb{Z}_n$
(7) Representations of permutation groups
(8) Symmetry groups
(9) Subgroups
(10) Equivalence relations and partitions
(11) Congruences and Euler's Phi Function

Computational problems you must be able to do

(1) Filling out a Cayley table of an operation

(2) Showing that a set is a group or a subgroup
(3) Working with various representations of permutation groups
(4) Finding a symmetry group of a shape in the plane
(5) Showing that a relation is an equivalence relation
(6) Finding equivalence classes
(7) Working with congruences
(8) Computing greatest common divisor using Euclidean algorithm
(9) Expressing (a, b) as a linear combination of a and b
(10) Finding $\phi(n)$ from prime factorization of n

How to study for the exam

The exam will be friendly to those who have studied carefully and followed all the instructions on this sheet. Most of the test questions will look familiar. You will be asked to repeat some definitions, state some theorems, and reproduce some proofs you have seen before. The exam will contain some exercises you have not seen before, but they will not comprise the bulk of the exam, and there will be no questions that only divine intervention will help you solve. You will do poorly if you fail to follow the advice on this preparation sheet.

(1) Read this worksheet thoroughly.
(2) Read and understand your class notes.
(3) Know how to do all the homework and quiz problems. The solutions are on our class conference.
(4) Go to office hours to ask questions.
(5) After you have done all of the above, start on the review problems below.

Review Problems (solutions will be provided later)

(1) Prove that, for all sets A, B, C, we have $(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C)$.

(2) Construct a Cayley table for the set $\{1, -1, i, -i\}$, where i is the complex number with $i^2 = -1$. The operation is multiplication.

(3) Consider a vector space V over $\mathbb{R}$, and let $L(V)$ be the collection of all linear transformations $T\colon V \to V$. Recall that $T\colon V \to V$ is a linear transformation if $T(cv + w) = cT(v) + T(w)$ for all $c \in \mathbb{R}$ and $v, w \in V$.
   (a) Prove that the identity function on V belongs to $L(V)$ and is the identity element for $(L(V), \circ)$.
   (b) Prove that composition $\circ$ is an operation on $L(V)$.

(4) For each of the following, determine with proof whether or not it is a group.
   (a) $(G, \cdot)$ where $G = \{2^m 3^n : m, n \in \mathbb{Z}\}$ and $\cdot$ is the usual multiplication on real numbers.
   (b) $(G, +)$ where $G = \{a + b\sqrt{2}\colon a, b \in \mathbb{Z}\}$ and $+$ is the usual addition of real numbers.

(5) Let G be a group and let $a \in G$ be a fixed element. Define the function $\lambda_a\colon G \to G$ given by $\lambda_a(g) = ag$ for all $g \in G$. Prove that λ_a is a bijection.

(6) Prove that A_n is nonabelian if $n > 3$. Write out the Cayley table for A_3.

(7) Determine with proof whether or not the set $\{\alpha \in S_4\colon \sigma(2) \neq 3\}$ is a subgroup of S_4.

(8) Write $(245)(1354)(125)$ and $(35)(123)(12)$ as (i) a cycle or a product of disjoint cycles in S_5; (ii) a product of transpositions in S_5. Decide if each is even or odd.

(9) Consider the set $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ endowed with a multiplication such that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$ and $ik = -j$.
   (a) Write out the Cayley table for Q_8 (you will have an 8×8 grid).
   (b) Find the identity element and list the inverse of each of the 8 elements.

We will assume that associativity holds. This $(Q_8, \cdot)$ is called the *quaternion group*.

(c) Find the center of Q_8.

(d) Show that, if G is a group, then there is an equivalence relation given by $a \sim b$ in G iff $a = gbg^{-1}$ for some $g \in G$. Find the equivalence classes with respect to $\sim$ if $G = Q_8$.

(10) Determine whether the following subsets H are subgroups of the given group G.

(a) Let $G = \mathbb{Q}$ and $H = \left\{ \frac{n}{2^m} : n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0} \right\}$.

(b) Let $G = M(\mathbb{R})$ and $H = \left\{ f \in M(\mathbb{R}) \colon f(2) = 0 \text{ and } \int_0^1 x f(x)\, dx = 0 \right\}$.

(c) Fix $n \in \mathbb{Z}_{\geq 0}$. Let $G = M(\mathbb{R})$ and

$$H = \{ f \in M(\mathbb{R}) \colon f(x) = c_0 + c_1 x + \cdots + c_n x^n \text{ for some } c_0, \ldots, c_n \in \mathbb{R} \text{ and for all } x \in \mathbb{R} \}.$$

(11) (a) Let $a \in \mathbb{R}^\times$. Prove that the matrix $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ belongs to the center of $\mathrm{GL}_2(\mathbb{R})$.

(b) Prove that, if a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to the center of $\mathrm{GL}_2(\mathbb{R})$, then $b = c = 0$ and $a = d$. Hint: such a matrix must commute with every element of $\mathrm{GL}_2(\mathbb{R})$, so in particular it commutes with $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. What equations would we then have?

(12) For each of the following, prove that the relation given is or is not an equivalence relation. If it is an equivalence relation, describe a complete set of equivalence classes.

(a) Let $x \sim y$ in $\mathbb{R}$ iff $xy \geq 0$.

(b) Let $x \sim y$ in $\mathbb{R}$ iff $|x| = |y|$.

(13) List all the elements of $(\mathbb{Z}_9, \oplus)$ and the inverse of each.

(14) Use the Euclidean algorithm to find $(69, 372)$ and $(792, 275)$.

(15) Using the Euclidean algorithm, show that 69 and 89 are relatively prime and write 1 as a linear combination of the two numbers.

(16) Let n be an integer greater than 2. Prove that $\phi(n)$ is an even integer.