Math 305, Fall 2007
First Midterm Exam, October 11, 2007
Solutions

Name: ______________________________ Student ID: ___________________________

Directions: Check that your test has 10 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (10 points) ______________
2. (8 points) ______________
3. (5 points) ______________
4. (8 points) ______________
5. (6 points) ______________
6. (5 points) ______________
7. (8 points) ______________
8. (5 points) ______________

Total (out of 55): ______________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

   (a) Give an example of a noncommutative relation $*$ on $\{1, 2\}$.

   Solution: For example, set $1 * 1 = 1$, $1 * 2 = 1$, $2 * 1 = 2$, $2 * 2 = 2$.

   (b) Give an example of an infinite abelian group. Be sure to include the operation.

   Solution: $(\mathbb{Z}, +)$.

   (c) Give an example of a finite nonabelian group. Be sure to include the operation.

   Solution: S_n with composition, $n \geq 3$.

   (d) Give an example of a symmetry group of an object in the plane.

   Solution: The dihedral group D_4 is the symmetry group of a square.

   (e) Give the number of elements of A_n, $n \geq 3$.

   Solution: $\frac{n!}{2}$.

2. (4 pts each)

(a) Define precisely what it means for a function $f\colon S \to T$ to be *injective* and *surjective*.

Solution: A function $f\colon S \to T$ is *injective* if, for all $a, b \in S$, $f(a) = f(b) \implies a = b$. A function $f\colon S \to T$ is *surjective* if, for every $t \in T$, there is $s \in S$ such that $f(s) = t$.

(b) Let $M_2(\mathbb{R})$ denote the set of 2×2 matrices with real coefficients and recall that the determinant of a 2×2 matrix is given by the formula $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. We can thus think of the determinant as a map $\det\colon M_2(\mathbb{R}) \to \mathbb{R}$. Show that this map is surjective but not injective.

Solution: Let $a \in \mathbb{R}$. Then $\det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$, so det is surjective. Consider the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A \neq B$ and $\det(A) = \det(B)$, so det is not injective.

3. (5 pts) Verify that the set $G = \{2^m 3^n \colon m, n \in \mathbb{Z}\}$ is a group with respect to multiplication.

Solution:

Associativity: Given $2^{m_1}3^{n_1}, 2^{m_2}3^{n_2}, 2^{m_3}3^{n_3} \in G$, we have

$$(2^{m_1}3^{n_1} \cdot 2^{m_2}3^{n_2}) \cdot 2^{m_3}3^{n_3} = 2^{m_1}3^{n_1} \cdot (2^{m_2}3^{n_2} \cdot 2^{m_3}3^{n_3})$$

because ordinary multiplication is associative.

Identity: Identity is $2^0 3^0 = 1$.

Inverses: Inverse of $2^m 3^n$ is $2^{-m}3^{-n}$.

4. (4 pts each) Let $(G, *)$ be a group and suppose that A and B are subgroups of G. Define a subset $A * B$ of G by the following:

$$A * B = \{a * b \colon a \in A,\, b \in B\}.$$

(a) Consider $G = S_3$ and let $A = \{(1), (12)\}$ and $B = \{(1), (13)\}$. List the elements of $A * B$ and prove that $A * B$ is not a subgroup of G.

Solution: The elements of $A * B$ are $\{e, (12), (13), (132)\}$. This subset does not contain (123), the inverse of (132), so it is not a subgroup.

(b) Suppose that $(G, *)$ is abelian. Prove that, for any subgroups A and B of G, the subset $A * B$ is a subgroup of G.

Solution: Let A and B be subgroups of G. Then $e = e * e$ belongs to $A * B$, so $A * B$ contains the identity and is not empty. Let $ab, cd \in A * B$ with $a, c \in A$ and $b, d \in B$. Then $(ab)(cd)^{-1} = abd^{-1}c^{-1} = ac^{-1}bd^{-1} \in A * B$. Hence $A * B$ is a subgroup by a criterion from problem 7.22 on Homework 3.

5. (3 pts each)

(a) Show that S_n is nonabelian for $n \geq 3$.

Solution: See class notes from 9/17 or problem 6.11 on Homework 3.

(b) Decide whether $(245)(1354)(125) \in S_5$ is an even or an odd permutation.

Solution: This is part of problem 8 on the review exercises handout.

6. (5 pts) Show that, if H is a subgroup of $\mathbb{Z}$, then it has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

Solution: See class notes from 9/18 (this is one direction of an if and only if statement).

7. (4 pts each)

(a) For G a group with operation $*$ and for $a \in G$ define the *centralizer* of a in G to be the set

$$C(a) = \{x \in g\colon a * x = x * x\}.$$

Show that $C(a)$ is a subgroup of G.

Solution: This is problem 7.23 from Homework 3.

(b) Find the centralizer of 2 in $\mathbb{Z}$.

Solution: Since $\mathbb{Z}$ is abelian, every element commutes with every other element. In particular, everything commutes with 2, so $C(2) = \mathbb{Z}$.

8. (5 pts) Let G be a group and define a relation $\sim$ on G by the following: for all $a, b \in G$, $a \sim b$ iff there is $g \in G$ such that $b = gag^{-1}$. Prove that $\sim$ is an equivalence relation on G.

Solution: This is part of problem 9(d) on the review exercises handout.