Math 305, Fall 2007
Second Midterm Exam, November 15, 2007

Name: ______________________________ Student ID: ___________________________

Directions: Check that your test has 9 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (10 points) _____________
2. (6 points) _____________
3. (5 points) _____________
4. (5 points) _____________
5. (5 points) _____________
6. (8 points) _____________
7. (7 points) _____________

Total (out of 48): _____________

Did you attend Prof. Ben Brubaker's talk on Nov. 5? Answer "yes" or "no": _____________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

(a) Give an example of a non-identity, non-zero homomorphism $\theta\colon \mathbb{Z}_{10} \to \mathbb{Z}_5$.

(b) Give an example of a non-identity automorphism $\theta\colon \mathbb{Z} \to \mathbb{Z}$.

(c) Compute the number of elements in $\mathbb{Z}_{100}/\langle[15]\rangle$.

(d) Give representatives of all the isomorphism classes of abelian groups of order 56.

(e) Give an example of an infinite group all of whose elements have finite order.

2. (2 pts each) In each of the following, the two groups are not isomorphic. Give a reason why.

(a) $\mathbb{Z}$ and $\mathbb{R}$.

(b) $\mathbb{Z}_4 \times \mathbb{Z}_2$ and D_4.

(c) $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

3. (5 pts) State and prove Lagrange's Theorem.

4. (5 pts) Let $\theta\colon G \to H$ be a homomorphism. Prove that, if A is an abelian subgroup of G, then $\theta(A)$ is an abelian subgroup of H. Note: You need to first prove that $\theta(A)$ is a subgroup of H.

5. (5 pts) Let $\theta\colon G \to H$ be a homomorphism and let $K = \operatorname{Ker}\theta$. Consider the map $\phi\colon G/K \to \theta(G)$ given by $Ka \mapsto \theta(a)$ (this is the map from First Isomorphism Theorem). Show this map is well-defined.

6. (4 pts each)

   (a) Show that $\langle(1\,2\,3)\rangle$ is a normal subgroup of S_3.

   (b) List the elements of $S_3/\langle(1\,2\,3)\rangle$ and identify this quotient as a more familiar group.

7. (7 pts) Let $\phi\colon \mathbb{Q} \to \mathbb{Q}$ be a homomorphism. Prove that there is some $q \in \mathbb{Q}$ such that $\phi(a) = qa$ for all $a \in \mathbb{Q}$. (Hint: At some point you should consider the fact that, if $n \in \mathbb{Z}_{\geq 1}$, then $\phi(1) = \phi\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = n\phi\left(\frac{1}{n}\right)$.)