Math 305, Fall 2007
Midterm 2 Preparation

Your second midterm is on Thursday, November 15. It is a closed-note, closed-book, open-brain, no-calculator 70-minute examination held during the regular class time. It will cover the material from the first midterm up to and including Section 23. A review session will take place on Wednesday, November 14, in class. Please come prepared with questions.

What you need to commit to memory

(1) You must know the definitions of the following.

(a) Order of an element
(b) Cyclic (sub)group
(c) Direct product of groups
(d) Coset
(e) Index of a subgroup
(f) Homomorphism
(g) Isomorphism
(h) Automorphism
(i) Kernel of a homomorphism
(j) Normal subgroup
(k) Quotient group
(l) An extension of a group by another
(m) Simple group

(2) You must know the statements and proofs of the following.

(a) Statement that the relation $a \sim b \Leftrightarrow ab^{-1} \in H$, where H is a subgroup of G, is an equivalence relation
(b) Lagrange's Theorem
(c) Statements that a group of prime order is cyclic and that it is generated by any of its elements
(d) Statements that if a group is abelian or cyclic, so is any group isomorphic to it
(e) Theorem with five parts about properties of homomorphisms ($\theta(e_G) = e_H$, $\theta(a^{-1}) = \theta(a)^{-1}$, etc.)
(f) Theorem that isomorphism is an equivalence relation on the collection of all groups
(g) Statement that any cyclic group G is isomorphic to $\mathbb{Z}_{|G|}$ (this includes the case that if the order of G is a prime p, then G is isomorphic to $\mathbb{Z}_p$)
(h) Fundamental Theorem of Finite Abelian Groups (statement only)
(i) Cayley's Theorem (statement only)
(j) Theorem that (a) Kernel of a homomorphism is a subgroup of the domain, and (b) Kernel is trivial iff homomorphism is one-to-one (and so if kernel is trivial, domain is isomorphic to its image under the homomorphism)
(k) Statement that G/N, where N is a normal subgroup of G, is a group (including the part that this only works for a normal subgroup due to the issue of something being well-defined; this is not in your notes since you were asked to read it on your own)
(l) First Isomorphism Theorem

Topics you must study

(1) Divisibility properties of the order of an element
(2) Cyclic groups or groups generated by more than one element
(3) Direct products
(4) Cosets
(5) Lagrange's Theorem and its consequences
(6) Homomorphisms, isomorphisms, automorphisms
(7) Consequences of the Fundamental Theorem of Finite Abelian Groups
(8) Normal subgroups and quotients by normal subgroups
(9) First Isomorphisms Theorem

Computational problems you must be able to do

(1) Verifying that a group is cyclic
(2) Finding a direct product of two groups
(3) Finding all cosets of a subgroup
(4) Computing the index of a subgroup
(5) Verifying that a map is a homomorphism, isomorphism, or automorphism
(6) Showing that two groups are isomorphic or not
(7) Finding all representatives of isomorphism classes of an abelian group of some finite order
(8) Finding the kernel of a homomorphism
(9) Verifying that a subgroup is normal
(10) Identifying a quotient group as isomorphic to something familiar
(11) Finding an extension of a group by another

How to study for the exam

The exam will be friendly to those who have studied carefully and followed all the instructions on this sheet. Most of the test questions will look familiar. You will be asked to repeat some definitions, state some theorems, and reproduce some proofs you have seen before. The exam will contain some exercises you have not seen before, but they will not comprise the bulk of the exam, and there will be no questions that only divine intervention will help you solve. You will do poorly if you fail to follow the advice on this preparation sheet.

(1) Read this worksheet thoroughly.
(2) Read and understand your class notes.
(3) Know how to do all the homework and quiz problems. The solutions are on our class conference.
(4) Go to office hours to ask questions.
(5) After you have done all of the above, start on the review problems below.

Review Problems (solutions will be provided later)

(1) (a) Disprove: For all finite groups G and all $m \in \mathbb{Z}_{\geq 1}$, if $m \mid |G|$, then there is $a \in G$ such that $o(a) = m$. (Hint: Look at the quaternions Q_8.)
  (b) Prove that the statement in (b) is true if $G = \mathbb{Z}_n$ for any $n \in \mathbb{Z}_{\geq 2}$.

(2) Construct the *subgroup lattices* (see page 90 in Durbin) for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and for A_4.

(3) Recall that a nontrivial subgroup H of G is *proper* if $H \neq G$. Construct a proper subgroup of $\mathbb{Q}$ that is not cyclic. Hint: consider the *dyadic rationals*, fractions whose denominators are powers of 2.

(4) Let G be a group. Prove that, if G has at least two elements of order 2, then G is not cyclic.

(5) Compute the cosets of H in G and the index $[G : H]$ for the following groups G and subgroups H.
  (a) $G = S_4$ and $H = S_3$
  (b) $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and $H = \langle [2] \rangle \times \langle [2] \rangle$
  (c) $G = \mathbb{Z} \times \mathbb{Z}_2$ and $H = \langle (1, [1]) \rangle$
  (d) $G = \mathbb{R} \times \mathbb{Z}$ and $H = \mathbb{R} \times \{0\}$

(6) (a) Consider $\phi \colon \mathbb{Z}_6 \to \mathbb{Z}_3$ given by $\phi([a]_6) = [a]_3$. Prove that ϕ is a well-defined homomorphism and compute $\ker \phi$.
  (b) Prove that $\phi \colon \mathbb{Z}_3 \to \mathbb{Z}_6$ given by $\phi([a]_3) = [a]_6$ is not well-defined.
  (c) Let $n, m \in \mathbb{Z}_{>1}$. Determine a necessary and sufficient condition required for the function $\phi \colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ to be well-defined and prove your claim. Your claim should be an iff statement.

(7) (a) Let the *dihedral group* of $2n$ elements, denoted by D_n, be the group generated by two elements ρ and σ satisfying the relations $\rho^n = \sigma^2 = e$ and $\sigma\rho = \rho^{-1}\sigma$. Thus the $2n$ elements of D_n are

$$D_n = \{e, \rho, \rho^2, ..., \rho^{n-1}, \rho\sigma, \rho^2\sigma, ..., \rho^{n-1}\sigma\}.$$

This group turns out to be the symmetry group of a regular n-gon (so D_4 is the group of symmetries of a square. Prove that A_4 and D_{12} are not isomorphic. (Hint: Look at elements of order 2.)

(b) Define $\mathrm{SL}_2(\mathbb{Z}_3)$ to be the set of 2×2 matrices A with entries in $\mathbb{Z}_3$ such that $\det A = 1$ (as computed in $\mathbb{Z}_3$); i.e. add and multiply mod 3). Prove that $\mathrm{SL}_2(\mathbb{Z}_3)$ is a group that is not isomorphic to S_4. You may assume that $\mathrm{SL}_2(\mathbb{Z}_3)$ has 24 elements. (Hint: Look at elements of order 6.)

(8) Prove that the following pairs of groups are not isomorphic.

(a) D_8 and Q_8

(b) $\mathbb{R}$ and $\mathbb{Z}$

(c) D_{12} and S_4

(d) $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}_3$

(9) List all the isomorphism class representatives of abelian groups of order 240.

(10) Determine (with proof) whether the following are automorphisms.

(a) Define $\phi\colon S_3 \to S_3$ by $\phi(\alpha) = \alpha^{-1}$.

(b) Define $\phi\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ by $\phi([a]) = [3a]$.

(11) Let G be a group and let H and K be normal subgroups with $H \cap K = \{e\}$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. (Hint: Show that $x^{-1}y^{-1}xy \in H \cap K$.)

(12) Let G be a group all of whose subgroups generated by a single element are normal. For $a, b \in G$, prove that there is $k \in \mathbb{Z}$ such that $ab = ba^k$.

(13) Let H be the subset of $M(\mathbb{R})$ consisting of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Decide if H is normal in $(M(\mathbb{R}), +)$.

(14) Find the order of $12\mathbb{Z} + 8$ in $\mathbb{Z}/12\mathbb{Z}$ and the order of $\mathrm{SL}_2(\mathbb{R})\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R})$.

(15) Compute the number of elements in $\mathbb{Z}_{100}/\langle[15]\rangle$.

(16) Let G be a group and recall that $Z(G)$ is the center of G. Also recall from class that $Z(G)$ is a normal subgroup of G, so that we may consider the quotient $G/Z(G)$. Prove that, if $G/Z(G)$ is cyclic, then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $Z(G)x$, show that every element of G can be written in the form $x^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.)

(17) Let $A \subseteq G$ be nonempty and define $N_G(A) = \{g \in G\colon gAg^{-1} = A\}$. Here $gAg^{-1} = \{gag^{-1}\colon a \in A\}$. We call $N_G(A)$ the *normalizer* of A in G.

(a) Prove that, for any nonempty $A \subseteq G$, the set $N_G(A)$ is a subgroup. Note that A does not have to be a subgroup.

(b) Prove that, if H is a subgroup of G, then it is a subgroup of $N_G(H)$.

(c) Let $C_G(H)$ be the set of elements of G which commute with all elements of H. Prove that, if H is a subgroup, then $C_G(H)$ is a subgroup of $N_G(H)$, but that equality does not always hold.

(d) Prove that, if H is a normal subgroup of G, then $N_G(H) = G$.

(e) Find the normalizer of each subgroup of S_3 and Q_8.

(18) Let $(G, \circ)$ be the group of all functions $\tau_{a,b}\colon \mathbb{R} \to \mathbb{R}$ given by $\tau_{a,b}(x) = ax + b$, where $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$. Let N be the subset $N = \{\tau_{1,b} \in G\colon b \in \mathbb{R}\}$. Prove that $N \lhd G$ and that $G/N \cong \mathbb{R}^\times$.