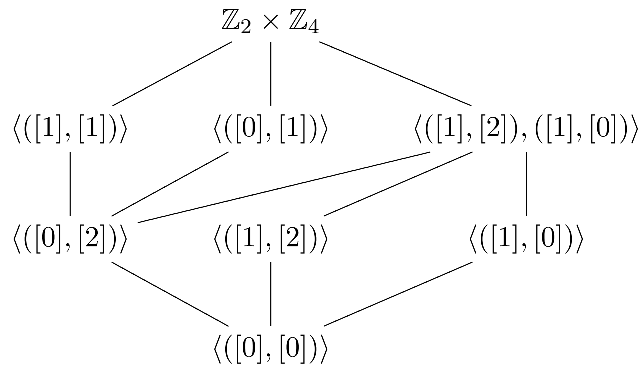
Math 305, Fall 2007
Midterm 2 Review Solutions

(1) (a) Disprove: For all finite groups G and all $m \in \mathbb{Z}_{\geq 1}$, if $m||G|$, then there is $a \in G$ such that $o(a) = m$. (Hint: Look at the quaternions Q_8.)
(b) Prove that the statement in (b) is true if $G = \mathbb{Z}_n$ for any $n \in \mathbb{Z}_{\geq 2}$.

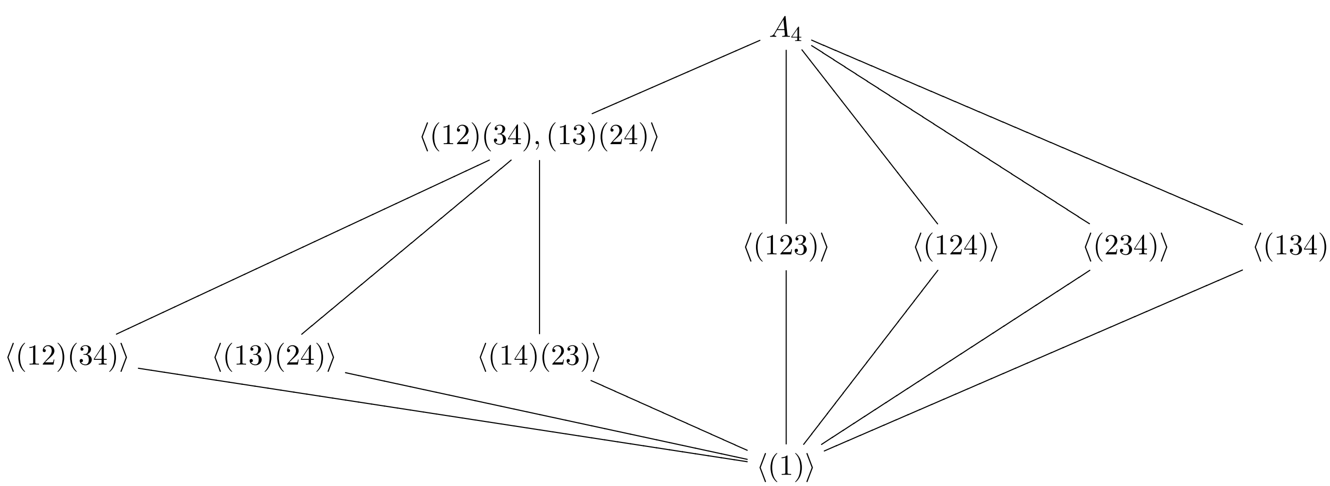
Solution:
(a) Let $G = Q_8$ and $m = 8$. Then clearly G is not cyclic, and hence for all $a \in G$ we have $o(a) < 8$.
(b) First claim is that, in a finite cyclic group $\langle a \rangle$ of order n, if $k \in \mathbb{Z}$, then $\langle a^k \rangle$ is a subgroup of order $n/(n,k)$ (think about this). Hence for the cyclic group $\mathbb{Z}_n$, since $[1]$ is a generator, we know that $[k]$ generates a subgroup of order $n/(n,k)$ for all $k \in \mathbb{Z}$. Let $m|n$. Then $d = n/m$ certainly divides n, so $(n,d) = d$. By the above, the element $[d]$ generates a subgroup of order $n/(n,d) = n/d = m$, as required.

(2) Construct the *subgroup lattices* (see page 90 in Durbin) for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and for A_4.

Solution: For $\mathbb{Z}_2 \times \mathbb{Z}_4$, we have the lattice



The lattice for A_4 is as



(3) Recall that a nontrivial subgroup H of G is *proper* if $H \neq G$. Construct a proper subgroup of $\mathbb{Q}$ that is not cyclic. Hint: consider the *dyadic rationals*, fractions whose denominators are powers of 2.

Solution: Consider the subset H of $\mathbb{Q}$ given by $\{m/2^n : m \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0}\}$. Certainly this subset is nonempty. Consider $a = m/2^n$ and $b = r/2^s$, where $m, r \in \mathbb{Z}$ and $r, s \in \mathbb{Z}_{\geq 0}$. Then $a - b = (2^s m - 2^n r)/2^{n+s}$, so $a - b$

belongs to H. Hence H is a subgroup. Suppose that $H = \langle m/2^n \rangle$, where $m \in \mathbb{Z}$ is odd and $n \in \mathbb{Z}_{\geq 0}$. Then clearly $m/2^{n+1}$ does not belong to $\langle m/2^n \rangle$ but does belong to H, a contradiction.

(4) Let G be a group. Prove that, if G has at least two elements of order 2, then G is not cyclic.

Solution: Suppose that G is cyclic. If G is infinite, then $G \cong \mathbb{Z}$, so G has no elements of order 2, a contradiction. If G is finite, then $G \cong \mathbb{Z}_n$ for some $n \in \mathbb{Z}_{\geq 2}$. If $[a]_n \in G$ has order two, then $[2a]_n = 0$. If n is odd, then the only $[a]_n$ satisfying this equation is $[a]_n = [0]_n$ which has order 1. If n is even, then the only $[a]_n$ satisfying this equation are $[a]_n = [0]_n$, which has order 1, or $[a]_n = [n/2]_n$, but then there is only one.

(5) Compute the cosets of H in G and the index $[G : H]$ for the following groups G and subgroups H.
 (a) $G = S_4$ and $H = S_3$
 (b) $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and $H = \langle [2] \rangle \times \langle [2] \rangle$
 (c) $G = \mathbb{Z} \times \mathbb{Z}_2$ and $H = \langle (1, [1]) \rangle$
 (d) $G = \mathbb{R} \times \mathbb{Z}$ and $H = \mathbb{R} \times \{0\}$

Solution:
 (a) Set of cosets is $\{H, H(14), H(24), H(34)\}$ and $[G\colon H] = 4$.
 (b) Set of cosets is $\{H, H\rho, H\rho^2, H\rho^3, H\rho^4\}$ and $[G\colon H] = 5$.
 (c) Set of cosets is $\{H, H + ([0], [1]), H + ([1], [0]), H + ([1], [1])\}$ and $[G\colon H] = 4$.
 (d) Set of cosets is $\{H, H + (1, [0])\}$ and $[G\colon H] = 2$.
 (e) Set of cosets is $\{\ldots, H + (0, -1), H, H + (0, 1), H + (0, 2), \ldots\}$ and $[G\colon H] = \infty$.

(6) (a) Consider $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_3$ given by $\phi([a]_6) = [a]_3$. Prove that ϕ is a well-defined homomorphism and compute $\ker \phi$.
 (b) Prove that $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_6$ given by $\phi([a]_3) = [a]_6$ is not well-defined.
 (c) Let $n, m \in \mathbb{Z}_{>1}$. Determine a necessary and sufficient condition required for the function $\phi\colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ to be well-defined and prove your claim. Your claim should be an iff statement.

Solution:
 (a) Let $[a]_6 = [b]_6$ in $\mathbb{Z}_6$. Then $6|(a - b)$. Hence $3|(a - b)$. so $[a]_3 = [b]_3$ in $\mathbb{Z}_3$, so $\phi([a]_6) = \phi([b]_6)$. Hence ϕ is well-defined. The kernel of ϕ is $\{[0]_6, [3]_6\}$.
 (b) Consider $[0]_3$ and $[3]_3$. We have $[0]_3 = [3]_3$ in $\mathbb{Z}_3$ but $[0]_6 \neq [3]_6$ in $\mathbb{Z}_6$. Hence $\phi([0]_3) \neq \phi([3]_3)$, so ϕ is not well-defined.
 (c) We claim that $\phi\colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ is well-defined iff $m|n$. First suppose that $m|n$. If $[a]_n = [b]_n$ in $\mathbb{Z}_n$, then $n|(a - b)$. Since $m|n$, we then have $m|(a - b)$, so $[a]_m = [b]_m$ in $\mathbb{Z}_m$. Hence $\phi([a]_n) = \phi([b]_n)$. Therefore ϕ is well-defined. Now suppose that m does not divide n. Consider $[n]_n$ and $[0]_n$ in $\mathbb{Z}_n$. Clearly $[n]_n = [0]_n$ in $\mathbb{Z}_n$. However, if $[n]_m = [0]_m$ in $\mathbb{Z}_m$, then $m|n-0$, a contradiction. Hence $[n]_m \neq [0]_m$ in $\mathbb{Z}_m$ so ϕ is not well-defined.

(7) (a) Let the *dihedral group* of $2n$ elements, denoted by D_n, be the group generated by two elements ρ and σ satisfying the relations $\rho^n = \sigma^2 = e$ and $\sigma\rho = \rho^{-1}\sigma$. Thus the $2n$ elements of D_n are

$$D_n = \{e, \rho, \rho^2, ..., \rho^{n-1}, \rho\sigma, \rho^2\sigma, ..., \rho^{n-1}\sigma\}.$$

This group turns out to be the symmetry group of a regular n-gon (so D_4 is the group of symmetries of a square. Prove that A_4 and D_{12} are not isomorphic. (Hint: Look at elements of order 2.)
 (b) Define $\mathrm{SL}_2(\mathbb{Z}_3)$ to be the set of 2×2 matrices A with entries in $\mathbb{Z}_3$ such that $\det A = 1$ (as computed in $\mathbb{Z}_3$); i.e. add and multiply mod 3). Prove that $\mathrm{SL}_2(\mathbb{Z}_3)$ is a group that is not isomorphic to S_4. You may assume that $\mathrm{SL}_2(\mathbb{Z}_3)$ has 24 elements. (Hint: Look at elements of order 6.)

Solution:
 (a) The only elements of order 2 in A_4 are (12)(34), (13)(24) and (14)(23). But there are seven elements of order 2 in D_{12}, namely σ, $\rho\sigma$, $\rho^2\sigma$, $\rho^3\sigma$, $\rho^4\sigma$, $\rho^5\sigma$ and ρ^3. •

(b) The group S_4 has no element of order 6. However the matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ has order 6 in $\mathrm{SL}_2(\mathbb{Z}_3)$.

(8) Prove that the following pairs of groups are not isomorphic.
(a) D_8 and Q_8
(b) $\mathbb{R}$ and $\mathbb{Z}$
(c) D_{12} and S_4
(d) $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}_3$

Solution:
(a) The group Q_8 has only one element of order 2, while D_8 has five elements of order 2.
(b) The group $\mathbb{Z}$ is cyclic but $\mathbb{R}$ is not.
(c) In D_{24} the element ρ has order 12, but in S_4 there are no elements of order 12.
(d) The group $\mathbb{Z} \times \mathbb{Z}_2$ has an element of order 2, namely $(0, [1])$. However, no element of $\mathbb{Z} \times \mathbb{Z}_3$ has order 2.

(9) List all the isomorphism class representatives of abelian groups of order 240.

Solution: Note that $240 = 2^4 \cdot 3 \cdot 5$. The abelian groups of order 240 are $\mathbb{Z}_{2^4} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(10) Determine (with proof) whether the following are automorphisms.
(a) Define $\phi\colon S_3 \to S_3$ by $\phi(\alpha) = \alpha^{-1}$.
(b) Define $\phi\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ by $\phi([a]) = [3a]$.

Solution:
(a) Let $x = (12)$ and $y = (13)$. Then $\phi(xy) = \phi((132)) = (123)$ and $\phi(x)\phi(y) = (12)(13) = (132)$, so ϕ is not a homomorphism and thus not an automorphism.
(b) Notice that $\phi([0]) = \phi([4]) = 0$, but $[0] \neq [4]$ in $\mathbb{Z}_{12}$, so ϕ is not injective. Hence it is not an automorphism.

(11) Let G be a group and let H and K be normal subgroups with $H \cap K = \{e\}$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. (Hint: Show that $x^{-1}y^{-1}xy \in H \cap K$.)

Solution: Let $x \in H$ and $y \in K$. Then $x^{-1}y^{-1}x$ lies in K, so $x^{-1}y^{-1}xy \in K$. Also $y^{-1}xy \in H$, so $x^{-1}y^{-1}xy \in H$. Therefore $x^{-1}y^{-1}xy \in H \cap K$. So $x^{-1}y^{-1}xy = e$. Hence $xy = yx$.

(12) Let G be a group all of whose subgroups generated by a single element are normal. For $a, b \in G$, prove that there is $k \in \mathbb{Z}$ such that $ab = ba^k$.

Solution: Let $a, b \in G$. Then $\langle a \rangle$ is normal in G. Hence $b^{-1}ab \in \langle a \rangle$, so $b^{-1}ab = a^k$ for some $k \in \mathbb{Z}$. Therefore we have $ab = ba^k$.

(13) Let H be the subset of $M(\mathbb{R})$ consisting of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Decide if H is normal in $(M(\mathbb{R}), +)$.

Solution: Since $(M(\mathbb{R}), +)$ is abelian, all subgroups are normal.

(14) Find the order of $12\mathbb{Z} + 8$ in $\mathbb{Z}/12\mathbb{Z}$ and the order of $\mathrm{SL}_2(\mathbb{R}) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R})$.

Solution: 3 and ∞.

(15) Compute the number of elements in $\mathbb{Z}_{100}/\langle [15] \rangle$.

Solution: Since the order of $[15]$ in $\mathbb{Z}_{100}$ is 20, the order of the quotient group is $|\mathbb{Z}_{100}|/|\langle [15] \rangle| = 100/20 = 5$.

(16) Let G be a group and recall that $Z(G)$ is the center of G. Also recall from class that $Z(G)$ is a normal subgroup of G, so that we may consider the quotient $G/Z(G)$. Prove that, if $G/Z(G)$ is cyclic, then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $Z(G)x$, show that every element of G can be written in the form $x^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.)

Solution: Let $g \in G$ and consider $Z(G)g \in G/Z(G)$. Since $G/Z(G)$ is cyclic, it is generated by some $Z(G)x$, so there is $k \in \mathbb{Z}$ such that $Z(G)g = (Z(G)x)^k = Z(G)x^k$. So $g = z_1 x^k$ for some $z_1 \in Z(G)$. Similarly if $h \in G$, then $h = z_2 x^j$ for some $z_2 \in Z(G)$ and $j \in \mathbb{Z}$. Therefore $gh = z_1 x^k z_2 x^j = z_1 z_2 x^{j+k} = z_2 x^k z_1 x^j = hg$, so G is abelian.

(17) Let $A \subseteq G$ be nonempty and define $N_G(A) = \{g \in G \colon gAg^{-1} = A\}$. Here $gAg^{-1} = \{gag^{-1} \colon a \in A\}$. We call $N_G(A)$ the *normalizer* of A in G.

(a) Prove that, for any nonempty $A \subseteq G$, the set $N_G(A)$ is a subgroup. Note that A does not have to be a subgroup.

(b) Prove that, if H is a subgroup of G, then it is a subgroup of $N_G(H)$.

(c) Let $C_G(H)$ be the set of elements of G which commute with all elements of H. Prove that, if H is a subgroup, then $C_G(H)$ is a subgroup of $N_G(H)$, but that equality does not always hold.

(d) Prove that, if H is a normal subgroup of G, then $N_G(H) = G$.

(e) Find the normalizer of each subgroup of S_3 and Q_8.

Solution:

(a) Let $g, h \in N_G(A)$. Then $gAg^{-1} = A$ and $hAh^{-1} = A$. Note that $h^{-1}Ah = A$ as well. So $h^{-1} \in N_G(A)$. Also $ghA(gh)^{-1} ghAh^{-1}g^{-1} = gAg^{-1} = A$, so $gh \in N_G(A)$, so $N_G(A)$ is closed. Certainly $eAe^{-1} = A$, so $N_G(A)$ has the identity. Therefore it is a subgroup of G.

(b) It suffices to show that $H \subseteq N_G(H)$. Let $h \in H$. Then $hH = H = Hh$, so $hHh^{-1} = H$, so $h \in N_G(H)$.

(c) Let $g \in C_G(H)$. Then g commutes with every element $h \in H$. In particular, we have $gH = Hg$, so $gHg^{-1} = H$. Therefore $g \in N_G(H)$. An example in which equality does not hold is $H = A_3$ and $G = S_3$.

(d) Let H be normal in G. Let $g \in G$. Then $gH = Hg$, so $gHg^{-1} = H$. Therefore $g \in N_G(H)$. So $G = N_G(H)$.

(e) In S_3, the normalizer of $\{e\}$, $\langle(123)\rangle$ and S_3 is S_3 itself. Each of the other subgroups is its own normalizer. In Q_8, the normalizer of every subgroup is Q_8.

(18) Let $(G, \circ)$ be the group of all functions $\tau_{a,b} \colon \mathbb{R} \to \mathbb{R}$ given by $\tau_{a,b}(x) = ax + b$, where $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$. Let N be the subset $N = \{\tau_{1,b} \in G \colon b \in \mathbb{R}\}$. Prove that $N \lhd G$ and that $G/N \cong \mathbb{R}^\times$.

Solution: Let $\phi \colon G \to \mathbb{R}^\times$ be given by $\phi(\tau_{a,b}) = a$ for all $\tau_{a,b} \in G$. It is not hard to show that this map is a well-defined homomorphism with kernel N. Also the image is $\mathbb{R}^\times$. Hence by the First Isomorphism Theorem, we have $G/N \cong \mathbb{R}^\times$.