Math 305, Fall 2007
Second Midterm Exam, November 15, 2007
Solutions

Name: ______________________________ Student ID: ___________________________

Directions: Check that your test has 9 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (10 points) _____________
2. (6 points) _____________
3. (5 points) _____________
4. (5 points) _____________
5. (5 points) _____________
6. (8 points) _____________
7. (7 points) _____________

Total (out of 48): _____________

Did you attend Prof. Ben Brubaker's talk on Nov. 5? Answer "yes" or "no":_____________

1. (2 pts each) Give brief answers to the following questions. No explanations are required.

   (a) Give an example of a non-identity, non-zero homomorphism $\theta\colon \mathbb{Z}_{10} \to \mathbb{Z}_5$.

   Solution: See problem 6 from review exercises.

   (b) Give an example of a non-identity automorphism $\theta\colon \mathbb{Z} \to \mathbb{Z}$.

   Solution: The homomorphism that sends 1 to -1 is an automorphism.

   (c) Compute the number of elements in $\mathbb{Z}_{100}/\langle[15]\rangle$.

   Solution: See problem 15 on review exercises.

   (d) Give representatives of all the isomorphism classes of abelian groups of order 56.

   Solution: Since $56 = 2^3 \cdot 7$, by Fundamental Theorem of Finite Abelian Groups we have that the representatives are

$$\mathbb{Z}_{56},\ \mathbb{Z}_2 \times \mathbb{Z}_{28},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}.$$

   (e) Give an example of an infinite group all of whose elements have finite order.

   Solution: See homework 7, problem 22.12.

2. (2 pts each) In each of the following, the two groups are not isomorphic. Give a reason why.

(a) $\mathbb{Z}$ and $\mathbb{R}$.

Solution: $\mathbb{Z}$ is cyclic and $\mathbb{R}$ is not.

(b) $\mathbb{Z}_4 \times \mathbb{Z}_2$ and D_4.

Solution: $\mathbb{Z}_4 \times \mathbb{Z}_2$ has 4 elements of order 4 $((1,0),(3,0),(1,1),(3,1))$ and D_4 has only 2 (ρ, ρ^3).

(c) $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: $\mathbb{Z}_4 \times \mathbb{Z}_2$ has an element of order 4 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ does not.

3. (5 pts) State and prove Lagrange's Theorem.

Solution: See class notes from October 15.

4. (5 pts) Let $\theta\colon G \to H$ be a homomorphism. Prove that, if A is an abelian subgroup of G, then $\theta(A)$ is an abelian subgroup of H. Note: You need to first prove that $\theta(A)$ is a subgroup of H.

   Solution: Proof that $\theta(A)$ is a subgroup was essentially on Quiz 5. The proof that this is an abelian subgroup is A is was essentially done in class on October 17.

5. (5 pts) Let $\theta\colon G \to H$ be a homomorphism and let $K = \operatorname{Ker}\theta$. Consider the map $\phi\colon G/K \to \theta(G)$ given by $Ka \mapsto \theta(a)$ (this is the map from First Isomorphism Theorem). Show this map is well-defined.

Solution: This was mentioned as something you should read on your own, was reiterated on the review handout, and was done in class on November 14 (during review). Also see first paragraph of proof of First Isomorphism Theorem on page 115.

6. (4 pts each)

(a) Show that $\langle(1\,2\,3)\rangle$ is a normal subgroup of S_3.

Solution: It was mentioned in class on October 29 that this part is easy to do by conjugating with all the elements of S_3 (which are not elements of $\langle(1\,2\,3)\rangle$) one by one. More precisely, we have

$$\langle(1\,2\,3)\rangle = \{(1), (1\,2\,3), (1\,3\,2)\}$$

and

$$\begin{aligned}
(1\ 2)^{-1}\langle(1\,2\,3)\rangle(1\ 2) &= (1\ 2)\langle(1\,2\,3)\rangle(1\ 2) = \langle(1\,2\,3)\rangle \\
(1\ 3)^{-1}\langle(1\,2\,3)\rangle(1\ 3) &= (1\ 3)\langle(1\,2\,3)\rangle(1\ 3) = \langle(1\,2\,3)\rangle \\
(2\ 3)^{-1}\langle(1\,2\,3)\rangle(2\ 3) &= (2\ 3)\langle(1\,2\,3)\rangle(2\ 3) = \langle(1\,2\,3)\rangle
\end{aligned}$$

(b) List the elements of $S_3/\langle(1\,2\,3)\rangle$ and identify this quotient as a more familiar group.

Solution: This was done in class on October 29.

7. (7 pts) Let $\phi\colon \mathbb{Q} \to \mathbb{Q}$ be a homomorphism. Prove that there is some $q \in \mathbb{Q}$ such that $\phi(a) = qa$ for all $a \in \mathbb{Q}$. (Hint: At some point you should consider the fact that, if $n \in \mathbb{Z}_{\geq 1}$, then $\phi(1) = \phi\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = n\phi\left(\frac{1}{n}\right)$.)

Solution: Let $q = \phi(1)$. Certainly $\phi(0) = 0 = q \cdot 0$. Let $n \in \mathbb{Z}_{>0}$. Then

$$q = \phi(1) = \phi\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = n\,\phi\left(\frac{1}{n}\right).$$

So $\phi\left(\frac{1}{n}\right) = q \cdot \frac{1}{n}$. Let $\frac{m}{n} \in \mathbb{Q}$ with $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Then

$$\phi\left(\frac{m}{n}\right) = \phi\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = m\,\phi\left(\frac{1}{n}\right) = q \cdot \frac{m}{n}.$$

Therefore we have proved the claim for all nonnegative rational numbers. Lastly, if $a < 0$, then $\phi(a) = -\phi(-a) = -(q)(-a) = qa$.