Math 305, Quiz 6 Solutions
October 31, 2007

Name:______________________________

(1) (5 pts) Prove that, if $\theta \colon G \to H$ is a homomorphism, then $\mathrm{Ker}(\theta)$ is a normal subgroup of G (you can assume it is a subgroup).

Solution: Let $n \in \mathrm{Ker}(\theta)$ and $g \in G$. Then

$$\theta(gng^{1}) = \theta(g)\theta(n)\theta(g^{-1}) = \theta(g)\theta(g^{-1}) = \theta(g)(\theta(g))^{-1} = e_H,$$

and so $gng^{1} \in \mathrm{Ker}(\theta)$. (First and third equalities in above are true because θ is a homomorphism and the second is true since n is in $\mathrm{Ker}(\theta)$.)

(2) (5 pts) Given homomorphisms $\alpha \colon G \to H$ and $\beta \colon H \to K$, show that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta \circ \alpha)$, but that equality need not hold.

Solution: If $a \in \mathrm{Ker}(\alpha)$, then $(\beta \circ \alpha)(a) = \beta(\alpha(a)) = \beta(e_H) = e_K$ (the last equality is true since β is a homomorphism). Thus $a \in \mathrm{Ker}(\beta \circ \alpha)$.

To show that equality need not hold, take for example $\alpha \colon \mathbb{Z} \to n\mathbb{Z}$ given by $\alpha(a) = na$ and $\beta \colon n\mathbb{Z} \to K$ given by $\beta(m) = e_K$ for all $m \in n\mathbb{Z}$ (here K is any group). Then $\mathrm{Ker}(\alpha) = \{0\}$ but $\mathrm{Ker}(\beta \circ \alpha) = \mathbb{Z}$.