Math 305, Quiz 8 Solutions
December 6, 2007

Name:______________________________

(1) (5 pts) Draw a Venn diagram for the sets *rings*, *commutative rings*, *unital commutative rings*, *integral domains* and *fields*. Insert in this diagram the following rings: $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Z}[\sqrt{3}]$, $M_2(\mathbb{C})$, $\mathbb{Z}[i]$, $\mathbb{Z}_5[x]$, $\mathbb{Z}_6[x]$, $2\mathbb{Z} \times 3\mathbb{Z}$, $5\mathbb{Z}$ and $M(\mathbb{R})$. (Recall that $M_2(\mathbb{C})$ is the ring of 2×2 matrices over $\mathbb{C}$ and $M(\mathbb{R})$ is the ring of functions from $\mathbb{R}$ to $\mathbb{R}$.)

Solution: The fields are $\mathbb{C}$ and $\mathbb{Q}(\sqrt{5})$. The integral domains that are not fields are $\mathbb{Z}$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[i]$ and $\mathbb{Z}_5[x]$. The unital commutative rings that are not integral domains are $\mathbb{Z}_6[x]$ and $M(\mathbb{R})$. The commutative rings that are not unital are $2\mathbb{Z} \times 3\mathbb{Z}$ and $5\mathbb{Z}$. The ring that is not commutative is $M_2(\mathbb{C})$.

(2) (5 pts) Decide (with proof) whether the map $\phi\colon \mathbb{C} \to M_2(\mathbb{R})$ given by $\phi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for all $a + bi \in \mathbb{C}$ is a ring homomorphism.

Solution: For all $a + bi, c + di \in \mathbb{C}$ we have $\phi((a + bi) + (c + di)) = \phi((a + c) + (b + d)i) = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \phi(a + bi) + \phi(c + di)$. We also have $\phi((a + bi)(c + di)) = \phi(ac - bd + (ad + bc)i) = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \phi(a+bi)\phi(c+di)$. Thus ϕ is a homomorphism.