Math 306 Topics in Algebra, Spring 2013
Takehome final exam

This exam is due Friday, May 17, by 4 pm. Late exams will not be graded and will receive an automatic zero. Please slide the exam under my door if I am not in my office. You can also scan your exam and email it to me as a single pdf file at any time before the deadline. You should work alone, and may use notes, homeworks (and everything proved there), and our textbook.

Also, please indicate the following clearly on the front page of your writeup:

(1) How many student seminars have you gone to this semester?
(2) Did you give a talk in the student seminar?
(3) Did you attend the colloquium on Euler given by Prof. William Dunham from Muhlenberg College?
(4) Did you attend the colloquium on braid groups given by Prof. Elisenda Grigsby from Boston College?
(5) Did you attend the colloquium on exotic spheres given by Prof. Mark Behrens from MIT?

(1) (10 pts) Suppose G is a finite group and H a subgroup such that $|H| \nmid [G:H]!$. Prove that H contains a nontrivial normal subgroup of G. (Hint: Define an action of G on the set of cosets of H in G and look at the kernel of the associated permutation representation.)

(2) (10 pts) Let G be a group of order pq where p, q are primes with $p < q$ and $q \not\equiv 1 \pmod{p}$. Use representation theory to show that G is abelian. (Hint: Use the fact that the number of one-dimensional representations of G divides $|G|$.)

(3) The following is the character table of a certain group G of order 60. The numbers in brackets are the numbers of elements in the conjugacy classes of representatives $g_i \in G$, χ_i is the character of an irreducible representation ϕ_i, $\alpha = (1+\sqrt{5})/2$, and $\beta = (1-\sqrt{5})/2$:

	$g_1 = e$ [1]	g_2 [20]	g_3 [15]	g_4 [12]	g_5 [12]
χ_1	1	1	1	1	1
χ_2	5	-1	1	0	0
χ_3	4	1	0	-1	-1
χ_4	3	0	-1	α	β
χ_5	a	b	c	d	e

(a) (3 pts) Determine the dimension of the representation ϕ_5.
(b) (6 pts) Determine χ_5.
(c) (6 pts) Show that G is a simple group.

(4) (7 pts/part) Let G be a finite group, let $g \in G$, and let χ be a character of G associated to an n-dimensional representation ϕ. Prove the following:
(a) If g has order 2, then $\chi(g) \in \mathbb{Z}$ and $\chi(g) \equiv \chi(e) \pmod{2}$.
(b) If g has order 4 and g is conjugate to g^{-1}, then $\chi(g) \in \mathbb{Z}$.

(5) (5 pts/part) Section 10.1, problem 8 (p. 344). (Note: In part (a), you can assume R is commutative since that is a part of the definition of an integral domain.)

(6) (a) (7 pts) Section 10.2, first part of problem 4 (p. 350)
(b) (7 pts) Section 10.2, second part of problem 4 (p. 350)
(c) (5 pts) Section 10.2, problem 5 (p. 350)

(7) (5 pts) Suppose M is an R-module and $f\colon M \to M$ is an R-module homomorphism satisfying $f \circ f = f$. Use a short exact sequence to show that $M = \ker f \oplus \operatorname{im} f$.