Math 306 Topics in Algebra, Spring 2013
Takehome midterm exam

This exam is due Thursday, April 4, by 5 pm. Late exams will not be graded and will receive an automatic zero. Please slide the exam under my door if I am not in my office. You should work alone, and may use notes, homework assignments (and everything proved there), and our textbook.

(1) (10 pts) An action of a group G on a set X is said to be *transitive* if there is only one orbit, i.e. given any $x_1, x_2 \in X$, there is a $g \in G$ such that $gx_1 = x_2$. Now suppose G is finite and the action $G \times X \to X$ is transitive. Choose $x \in X$ and let $H = \mathrm{Stab}_G(x)$ (the stabilizer of x). Show that $|X| = |G/H|$. Then deduce that $|G| = |X| \cdot |H|$. Thus G can only act transitively on a set which is finite and whose order divides the order of G.

(2) (7 pts) Section 4.5, problem 9 (p. 146).

(3) (10 pts) Let p and q be distinct odd primes. Show that any group G of order p^3q is not simple.

(4) (7 pts) Let σ (rotation by $\pi/2$) and ρ (reflection over the x-axis) be the usual generators of the dihedral group D_4. Define a representation $\phi\colon D_4 \to GL_2(\mathbb{C})$ by

$$\phi(\sigma^k) = \begin{pmatrix} i^k & 0 \\ 0 & (-i)^k \end{pmatrix}, \qquad \phi(\rho\sigma^k) = \begin{pmatrix} 0 & (-i)^k \\ i^k & 0 \end{pmatrix}.$$

Show that this is indeed a representation and prove that it is irreducible.

(5) (10 pts) Suppose $\phi\colon G \to GL(V)$ is equivalent to a decomposable representation. Show that ϕ is decomposable. (We stated this in class but did not prove it.)

(6) (5 pts/part)

(a) Let G be a finite abelian group and $\phi\colon G \to GL_n(\mathbb{C})$ a representation. Show that there exists an invertible matrix T such that $T^{-1}\phi_g T$ is diagonal for all $g \in G$ (so T is independent of g).

(b) Let A be an $n \times n$ matrix of finite order, i.e. $A^k = I$ for some positive integer k. Show that A is diagonalizable, i.e. it is similar to a diagonal matrix. (This is an important theorem in linear algebra.)

(7) (10 pts) We say a representation is *faithful* if the corresponding group action is. Show that, if G is a finite group and $\phi\colon G \to GL(V)$ is a faithful irreducible complex representation, then $Z(G)$ (the center of G) is cyclic. (Hint: Show that $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$.)