Math 306 Topics in Algebra, Spring 2013
Homework 1, due Friday, February 8

(1) (4 pts) Compute the number of elements in $\mathbb{Z}_{100}/\langle[15]\rangle$.

(2) (4 pts) Compute the cosets of S_3 in S_4 and find the index $[S_4 : S_3]$

(3) (4 pts/part)
   (a) Show that S_n is nonabelian for $n \geq 3$.
   (b) Decide whether $(245)(1354)(125) \in S_5$ is an even or an odd permutation.

(4) (4 pts/part)
   (a) For G a group with operation $*$ and for $a \in G$ define the *centralizer* of a in G to be the set
$$C(a) = \{x \in g\colon a * x = x * a\}.$$
   Show that $C(a)$ is a subgroup of G.
   (b) Find the centralizer of 2 in $\mathbb{Z}$.

(5) (5 pts) Let G be a group. Prove that, if G has at least two elements of order 2, then G is not cyclic.

(6) (a) (4 pts) Consider $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_3$ given by $\phi([a]_6) = [a]_3$. Prove that ϕ is a well-defined homomorphism and compute $\ker\phi$.
   (b) (4 pts) Prove that $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_6$ given by $\phi([a]_3) = [a]_6$ is not well-defined.
   (c) (5 pts) Let $n, m \in \mathbb{Z}_{>1}$. Determine a necessary and sufficient condition required for the function $\phi\colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ to be well-defined and prove your claim. Your claim should be an iff statement.

(7) (5 pts) Recall that the *dihedral group* of $2n$ elements, denoted by D_n, is the group generated by two elements ρ and σ satisfying the relations $\rho^n = \sigma^2 = e$ and $\sigma\rho = \rho^{-1}\sigma$. Thus the $2n$ elements of D_n are
$$D_n = \{e, \rho, \rho^2, ..., \rho^{n-1}, \rho\sigma, \rho^2\sigma, ..., \rho^{n-1}\sigma\}.$$
This group is the symmetry group of a regular n-gon (so D_4 is the group of symmetries of a square). Prove that A_4 and D_6 are not isomorphic. (Hint: Look at elements of order 6.)

(8) (5 pts) List all the (isomorphism class representatives of) abelian groups of order 240.

(9) (5 pts) Let G be a group and recall that $Z(G)$ is the center of G (look up the definition if you don't remember). Also recall that $Z(G)$ is a normal subgroup of G, so that we may consider the quotient $G/Z(G)$. Prove that, if $G/Z(G)$ is cyclic, then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $Z(G)x$, show that every element of G can be written in the form $x^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.)