# Math 306 Topics in Algebra, Spring 2013
## Homework 10, due Thursday, May 9, by 10am

**Note:** This is the last homework. Note that it is due in the morning and *cannot be turned in late*.

(1) (5 pts) Show that the sequence

$$C = \left(\mathbb{Z}/8\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{d_1} \mathbb{Z}/4\mathbb{Z}\right),$$

where each $d_i$ is multiplication by 4, is a chain complex and compute its homology (where that makes sense).

(2) (7 pts) Prove the *Splitting Lemma*: For a short exact sequence of modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the following are equivalent:

(a) There exists a map $f'\colon B \to A$ such that $f' \circ f$ is the identity on $A$.
(b) There exists a map $g'\colon C \to B$ such that $g \circ g'$ is the identity on $C$.
(c) $B \cong A \oplus C$ (so this short exact sequence can be replaced by the short exact sequence of the form $0 \to A \to A \oplus C \to C \to 0$).

A sequence that satisfies these conditions is called *split exact*.

(3) (5 pts) Recall the *Short Five Lemma* from lecture: Suppose the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0
\end{array}$$

of modules commutes and the rows are exact. Then

(a) $\alpha, \gamma$ injections $\Longrightarrow \beta$ is an injection;
(b) $\alpha, \gamma$ surjections $\Longrightarrow \beta$ is a surjection;
(c) $\alpha, \gamma$ isomorphism $\Longrightarrow \beta$ is an isomorphism;

Part (a) was proved in lecture. Prove part (b). (Note that part (c) follows from (a) and (b) immediately.)

(4) (5 pts) Section 10.5, problem 3 (p. 403)

(5) (5 pts) Section 10.5, problem 7(a) (p. 404)

(6) (5 pts) Section 10.5, problem 14(a) (p. 404)

(7) (5 pts) Section 10.4, problem 2 (p. 375)