Math 306 Topics in Algebra, Spring 2013
Homework 2, due Friday, February 15

(1) (5 pts/part)
  (a) Section 1.7, problem 5 (p. 44). (The kernel of an action is defined on page 43.)
  (b) An action of a group G on a set S is said to be *faithful* if for any $g \in G$, $g \neq e$, there exists an $s \in S$ such that $gs \neq s$. Show that an action is faithful if and only if the associated homomomorphism $\phi\colon G \to \mathrm{Sym}(S)$ has trivial kernel.

(2) (5 pts) Recall that a group G acts on a set S *transitively* if, for every $s, s' \in S$, there exists a $h \in G$ such that $hs' = s$. Also recall that the *stabilizer* of an element $s \in S$ is the subgroup $G_s = \{g \in G \mid gs = s\}$. Show that if G acts transitively on S, then all stabilizers are conjugate.

(3) (4 pts/part) The group $G = \langle (123)(45) \rangle$ is of order 6 and it acts on the set $\{1, 2, 3, 4, 5\}$ (by permuting it).
  (a) Determine $\mathrm{Orb}(k)$ for $1 \leq k \leq 5$.
  (b) Determine G_k for $1 \leq k \leq 5$.
  (c) Use parts (a) and (b) to verify that $|\,\mathrm{Orb}(k)| = |G|/|G_k|$ for $1 \leq k \leq 5$.

(4) This problem is about normalizers, which we used in the proof of the Extended Sylow's Theorem.
  (a) (4 pts) Show that, for any nonempty subset S of a group G, $N_G(S)$ is a subgroup of G.
  (b) (4 pts) Show that, for any $S \subset G$, $Z(G)$ (the center of G) is a subgroup of $N_G(S)$.
  (c) (5 pts) Let $G = S_3$. Show that the normalizer of $\{1, (123), (132)\}$ is all of S_3.
  (d) (5 pts) Let S be the set of all subsets of G. There is an action of G on S given by conjugation. Show that, for $T \in S$,

$$G_T = N_G(T).$$

  (Here G_T is the stabilizer of $T \in S$.)

(5) (4 pts/part) The last part of this problem was used in the example of the classification of groups of order 12. Let H and K be subgroups of a group G. Show the following.
  (a) If $H \cap K = \{1\}$, the product map

$$\begin{aligned} p\colon H \times K &\longrightarrow G \\ (h, k) &\longmapsto hk \end{aligned}$$

  is injective and the image is the subset HK of G.
  (b) If either H or K is a normal subgroup of G, then the product sets HK and KH are equal, and HK is a subgroup of G.
  (c) If H and K are normal, $H \cap K = \{1\}$, and $HK = G$ (as a set), then G is isomorphic to the product group $H \times K$.

(6) (3 pts) What are the orders of the Sylow p-subgroups of a group of order 700?