Math 306 Topics in Algebra, Spring 2013
Homework 4, due Friday, March 1

(1) (4 pts) Show that any abelian group can be made into a ring without unit by declaring the product of any two elements to be the identity in G.

(2) (5 pts) Show that $\mathbb{Z}_n$ is a field if and only if n is a prime (in which case we denote it by $\mathbb{F}_n$).

(3) (5 pts) Show that $GL_n(F)$, the set of all $n \times n$ matrices with non-zero determinant (where the determinant is computed the same way as in the case $F = \mathbb{R}$), is a group. You may use facts about determinant you know from linear algebra.

(4) Let $\mathcal{M}_{m\times n}$ be the additive group of all $m \times n$ matrices with coefficients in a field $\mathbb{F}$.
   (a) (5 pts) Show $\mathcal{M}_{m\times n}$ is a vector space over $\mathbb{F}$.
   (b) (4 pts) Find a basis and dimension of this space.
   (c) (5 pts) Show $\mathcal{M}_{m\times n}$ is isomorphic to $\mathbb{F}^{mn}$.

(5) (5 pts) Suppose B and B' are two bases for an n-dimensional vector space V and suppose $f\colon V \to V$ is an endomorphism. Let $T_{B,B}$ and $T_{B',B'}$ be matrix representations of f with respect to the two bases. Show that $T_{B,B}$ and $T_{B',B'}$ are conjugate, i.e. that there exists an $n \times n$ matrix A such that $AT_{B,B}A^{-1} = T_{B',B'}$.

(6) (4 pts/part)
   (a) Show that $\mathbb{F}$ is a one-dimensional vector space over itself.
   (b) Show that $GL(\mathbb{F}) \cong \mathbb{F}^{\times}$.

(7) (a) (3 pts) Show that, if $T \in \mathrm{End}(V)$ fixes vectors $v_1, ..., v_n$, i.e. $T(v_i) = v_i$, then it fixes the subspace generated by those vectors. In particular, $\mathrm{Span}\{v_1, ..., v_n\}$ is an invariant subspace for T.
   (b) (3 pts) Suppose $T \in \mathrm{End}(V)$. Show that the kernel of T (i.e. the nullspace of T) and the range of T are invariant subspaces of T.

(8) (5 pts) Suppose $T \in \mathrm{End}(V)$. Let λ be an eigenvalue of T and let V_λ be the eigenspace corresponding to λ (eigenvectors and eigenvalues are defined the same way for matrices over general field $\mathbb{F}$ as over $\mathbb{R}$). Show that V_λ is a T-invariant subspace of V.