Math 306 Topics in Algebra, Spring 2013
Homework 5, due Friday, March 8

Note: In the problems where you are asked to verify that something is a representation, don't forget to say why the image of ϕ is indeed in $GL(V)$, i.e. why ϕ_g is an automorphism of V for all $g \in G$. Usually a quick explanation will suffice.

(1) (5 pts) Prove that, for $A, B \in GL_n(F)$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Deduce that, if A and B are conjugate, they have the same trace.

(2) Let W_1 and W_2 be subspaces of a vector space V.
   (a) (3 pts) Show that W_1 and W_2 are independent if an only if $W_1 \cap W_2 = \{0\}$.
   (b) (5 pts) Show that there is an isomorphism $W_1 \times W_2 \cong W_1 \oplus W_2$.

(3) (a) (2 pts) Show that $\phi\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}^\times$ given by $\phi(m) = i^m$ is a (one-dimensional) representation.
   (b) (2 pts) More generally, show that $\phi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ given by $\phi(m) = e^{2\pi i m/n}$ is a (one-dimensional) representation.
   (c) (3 pts) Why does part (b) generalize part (a)?

(4) (5 pts) Let $V = \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$. Show that, for each $1 \le m \le n-1$, there is a representation $\mathbb{Z}/n\mathbb{Z} \to GL(V)$ given by

$$1 \longmapsto \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}$$

(5) This problem studies the *regular representation* of a group. Let G be a finite group or order n and F a field. Define $F[G]$ be the n-dimensional vector space generated by the elements of G. In other words, $F[G]$ is a vector space whose basis is G and whose elements are formal linear combinations of elements of G with coefficients in F. Now define the *left regular representation of* G by

$$L\colon G \longrightarrow GL(F[G])$$
$$g \longmapsto L_g(v) = gv$$

The multiplication here means the following: Since $v = \sum a_i g_i$, let $gv = g\sum a_i g_i = \sum a_i(gg_i)$. Similarly define the *right* regular representation $R\colon G \to GL(F[G])$ by $R_g(v) = vg^{-1}$.
   (a) (5 pts) Show that L is a representation, and then briefly say what changes for R.
   (b) (3 pts) Recall that representations correspond to linear actions. What actions to R and L correspond to?