Math 306 Topics in Algebra, Spring 2013
Homework 6, due Friday, March 15

(1) (5 pts) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternions (remind yourselves of the relations in this group). Construct a 2-dimensional representation of Q_8 over $\mathbb{C}$ (and show that it is really a representation).

(2) (4 pts) Restate the definition of the equivalence of representations in terms of group actions.

(3) (5 pts/part)

(a) Suppose representation $\phi\colon G \to GL(V)$ is decomposable, i.e. there exist nontrivial subspaces W_1, W_2 of V that are G-invariant and $V = W_1 \oplus W_2$. Show that a representation $\phi\colon G \to GL(V)$ is equivalent to the representation $\phi|_{W_1} \oplus \phi|_{W_2}$.

(b) Suppose G is generated by elements $g_1, ..., g_k$ and suppose $\phi\colon G \to GL(V)$ is a representation. Show that, if $\phi_{g_i} = \phi'_{g_i}|_{W_1} \oplus \phi''_{g_i}|_{W_2}$ for some subspaces W_1, W_2 satisfying $V = W_1 \oplus W_2$, then $\phi_g = \phi'_g|_{W_1} \oplus \phi''_g|_{W_2}$ for all $g \in G$.

(4) (5 pts) Suppose that $\phi\colon G \to GL(V)$ is an n-dimensional representation. Define $\phi^*\colon G \to GL(V)$ by $\phi^*_g = \phi^T_{g^{-1}}$, where T means the transpose matrix. Show that ϕ^* also defines an n-dimensional representation.

(5) (5 pts) Find four irreducible representations of $\mathbb{Z}_4$ over $\mathbb{C}$ (thought of as a 1-dimensional vector space over itself).

(6) (5 pts) In class, we had two 2-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$, ϕ and ψ, that we showed were equivalent. Show that neither of them is irreducible.

(7) (a) (4 pts) Show that $\mathbb{Z}_3 = \{e, x, x^2\}$ has a 3-dimensional representation ϕ over F given by

$$\phi_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \phi_{x^2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) (4 pts) Is ϕ equivalent to ϕ^*? Justify your answer. (See above for the definition of ϕ^*.)

(c) (4 pts) Show that the representation from part (a) is not irreducible by finding a 1-dimensional subrepresentation.

(d) (5 pts) Can you find more than one of these 1-dimensional subrepresentations? (Hint: The answer depends on F.)