Math 306 Topics in Algebra, Spring 2013
Homework 7, due Friday, April 12

(1) (5 pts) Let G be a finite group. Show that the function

$$\mathbb{C}[G] \times \mathbb{C}[G] \longrightarrow \mathbb{C}$$
$$(f_1, f_2) \longmapsto \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

defines an inner product on $\mathbb{C}[G]$.

(2) Instead of taking the trace of ϕ_g to define the character, one might try to do the same by taking the determinant of ϕ_g instead. This problem shows that this is not as useful since such a character would tell us nothing about non-abelian simple groups (and these are important).

For ϕ a representation of a finite group G, define a function

$$\det \phi \colon G \longrightarrow \mathbb{C}^\times$$
$$g \longmapsto (\det \phi)(g) = \det(\phi_g)$$

(a) (4 pts) Show $\det \phi$ is a representatation (and hence it's a character since characters are same as representations for one-dimensional representations).
(b) (5 pts) Show that if G is a non-abelian simple group, then $\det \phi$ is the trivial character.

(3) (4 pts) Show that $\chi_{\phi \oplus \psi} = \chi_\phi + \chi_\psi$.

(4) (5 pts/part)
(a) Suppose A is a matrix over $\mathbb{C}$ of finite order, i.e. $A^n = I$ for some positive integer n. Show that the eigenvalues λ_i of A are the nth roots of unity, namely they satisfy $\lambda_i^n = 1$. (Hint: Use the result that A is diagonalizable, which in turn follows from fact that for a representation of a finite group G, there exists a matrix T such that $T\phi_g T^{-1}$ is diagonal for all $g \in G$. Then look up what diagonalizability has to do with eigenvalues.)
(b) Prove that for an irreducible representation of a finite group G,

$$\chi_\phi(g) = \lambda_1 + \cdots + \lambda_d,$$

where λ_i are the eigenvalues of ϕ_g and d is the dimension of ϕ.
(c) Show that, if a complex number ω is a root of unity, then $\omega^{-1} = \overline{\omega}$.
(d) Prove that $\chi_\phi(g^{-1}) = \overline{\chi_\phi(g)}$.

(5) (5 pts) Recall that, in the proof of the theorem that a representation ϕ is irreducible iff $\langle \chi_\phi, \chi_\phi \rangle = 1$, we assumed $\phi \sim m_1\phi_1 \oplus \cdots \oplus m_s\phi_s$ and then claimed that

$$\langle \chi_\phi, \chi_\phi \rangle = m_1^2 + \cdots + m_s^2.$$

Show that this equation indeed holds.

(6) (a) (4 pts) Show that a finite group G is abelian if and only if it has $|G|$ irreducible representations (over $\mathbb{C}$).
(b) (5 pts) Use part (a) to show $\mathbb{Z}/n\mathbb{Z}$ is abelian. Do this without using that $\mathbb{Z}/n\mathbb{Z}$ has n conjugacy classes.

(7) This problem explores the regular representation over $\mathbb{C}$ and the associated character. The formula you will in part in part (d) is an important application of representation theory to group theory.

Recall from an earlier homework that the (left) regular representation of a finite group G is given by

$$L \colon G \longrightarrow GL(F[G])$$
$$g \longmapsto L_g(v) = gv.$$

(We also defined it in class using the dual vector space of $F[G]$, but for this problem, we'll stick to the original definition above.)

(a) (7 pts) Prove that the character of the regular representation is given by

$$\chi_L \colon G \longrightarrow \mathbb{C}$$
$$g \longmapsto \chi_L(g) = \begin{cases} |G|, & g = 1, \\ 0, & g \neq 1. \end{cases}$$

(b) (5 pts) We observed in class that every representation ϕ of G breaks up as

$$\phi \sim m_1\phi_1 \oplus \cdots \oplus m_s\phi_s,$$

where $\phi_1, ..., \phi_s$ is the complete set of irreducible representations of G (some of the m_i's might be zero). Prove that $\langle \chi_\phi, \chi_{\phi_i} \rangle = m_i$. (Hint: Use an earlier exercise and the linearity of the inner product.)

(c) (7 pts) Suppose d_i are the degrees of the irreducible representations ϕ_i of G. Show that if $\phi = L_g$, the m_i's from the previous part are precisely the degrees d_i. In other words, show that the decomposition

$$L \sim d_1\phi_1 \oplus \cdots \oplus d_s\phi_s$$

holds.

(d) (4 pts) Prove that

$$|G| = d_1^2 + \cdots + d_s^2.$$