Math 306 Topics in Algebra, Spring 2013
Homework 9, due Friday, April 26

(1) (5 pts) Show that, when R is a commutative ring and M is an R-module, $\mathrm{Hom}_R(M, M)$ is an R-algebra.

(2) (5 pts) Section 10.2, problem 1 (p. 350).

(3) (7 pts) Section 10.2, problem 9 (p. 350).

(4) (5 pts) Section 10.3, problem 4 (p. 356).

(5) (4 pts) Show that the sequence of modules

$$0 \longrightarrow A \xrightarrow{i} A \oplus C \xrightarrow{p} C \longrightarrow 0,$$

where i and p are the canonical inclusion and projection, is exact.

(6) (5 pts) Suppose $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4$ is exact. For $1 \leq k \leq 4$, set

$$C_k = \ker(A_k \longrightarrow A_{k+1}) = \mathrm{im}(A_{k-1} \longrightarrow A_k) = \mathrm{coker}(A_{k-2} \longrightarrow A_{k-1}).$$

(Note that, depending on k, some of these equivalent expressions for C_k may not make sense. For example, to define C_1, you have to use $C_1 = \ker(A_1 \to A_2)$ since the other two formulations would be in terms of A_0 and A_{-1} which we do not have. Similarly for C_5, you have to use $C_5 = \mathrm{coker}(C_3 \to C_4)$. Show that the sequences

$$0 \longrightarrow C_k \longrightarrow A_k \longrightarrow C_{k+1} \longrightarrow 0$$

are exact (you will have to define the maps as well). This therefore gives an example of how an exact sequence can be broken into (and spliced from) short exact sequences (the picture of how this works was drawn in class).

(7) (4 pts/part) For this problem, recall that by an extension we mean the entire exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

so that if there are two sequences with the same modules but different homomorphisms between them, we consider those extensions to be different.

(a) Show that any extension of C by A has $|C| \cdot |A|$ elements (sometimes this number is infinity).

(b) How many inequivalent extensions of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ are there? How about extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$?

(c) If p is a prime, show that there are exactly p non-isomorphic abelian extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$: the split extension and the extensions

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where p is multiplication by p and i is the multiplication by i for $1 \leq i \leq p-1$.