Math 306 Topics in Algebra, Spring 2013
Homework 1 Solutions

(1) (4 pts) Compute the number of elements in $\mathbb{Z}_{100}/\langle[15]\rangle$.

Solution: Since the order of $[15]$ in $\mathbb{Z}_{100}$ is 20, the order of the quotient group is $|\mathbb{Z}_{100}|/|\langle[15]\rangle| = 100/20 = 5$.

(2) (4 pts) Compute the cosets of S_3 in S_4 and find the index $[S_4 : S_3]$

Solution: Set of cosets is $\{S_3, S_3(14), S_3(24), S_3(34)\}$ and $[S_4 \colon S_3] = 4$.

(3) (4 pts/part)
(a) Show that S_n is nonabelian for $n \geq 3$.
(b) Decide whether $(245)(1354)(125) \in S_5$ is an even or an odd permutation.

Solution:
(a) Cycles (12) and (13) do not commute since $(12)(13) = (132)$ and $(13)(12) = (123)$.
(b) Since $(245) = (25)(24)$, $(1354) = (14)(15)(13)$, and $(125) = (15)(12)$, the total number of transpositions is 7, so the permutation is odd. (Alternatively, $(245)(1354)(125) = (14)(253) = (14)(53)(23)$.)

(4) (4 pts/part)
(a) For G a group with operation $*$ and for $a \in G$ define the *centralizer* of a in G to be the set

$$C(a) = \{x \in g \colon a * x = x * a\}.$$

Show that $C(a)$ is a subgroup of G.
(b) Find the centralizer of 2 in $\mathbb{Z}$.

Solution:
(a) Since $a * e = e * a$, $e \in C(a)$ and thus $C(a) \neq \emptyset$. If $x, y \in C(a)$, then

$$a * (x * y) = (a * x) * y = (x * a) * y = x * (a * y) = x * (y * a) = (x * y) * a$$

and so $x * y \in C(a)$. Finally, if $x \in C(a)$, then $a * x = x * a$, so

$$x^{-1} * a = x^{-1} * a * (x * x^{-1}) = x^{-1} * (a * x) * x^{-1} = x^{-1} * (x * a) * x^{-1} = (x^{-1} * x) * a * x^{-1} = a * x^{-1}$$

and so $x^{-1} \in C(a)$. Thus $C(a)$ is a subgroup.
(b) Since $\mathbb{Z}$ is abelian, every element commutes with every other element. In particular, everything commutes with 2, so $C(2) = \mathbb{Z}$.

(5) (5 pts) Let G be a group. Prove that, if G has at least two elements of order 2, then G is not cyclic.

Solution: Suppose that G is cyclic. If G is infinite, then $G \cong \mathbb{Z}$, so G has no elements of order 2, a contradiction. If G is finite, then $G \cong \mathbb{Z}_n$ for some $n \in \mathbb{Z}_{\geq 2}$. If $[a]_n \in G$ has order two, then $[2a]_n = 0$. If n is odd, then the only $[a]_n$ satisfying this equation is $[a]_n = [0]_n$ which has order 1. If n is even, then the only $[a]_n$ satisfying this equation are $[a]_n = [0]_n$, which has order 1, or $[a]_n = [n/2]_n$, but then there is only one.

(6) (a) (4 pts) Consider $\phi \colon \mathbb{Z}_6 \to \mathbb{Z}_3$ given by $\phi([a]_6) = [a]_3$. Prove that ϕ is a well-defined homomorphism and compute $\ker \phi$.
(b) (4 pts) Prove that $\phi \colon \mathbb{Z}_3 \to \mathbb{Z}_6$ given by $\phi([a]_3) = [a]_6$ is not well-defined.
(c) (5 pts) Let $n, m \in \mathbb{Z}_{>1}$. Determine a necessary and sufficient condition required for the function $\phi \colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ to be well-defined and prove your claim. Your claim should be an iff statement.

Solution:
(a) Let $[a]_6 = [b]_6$ in $\mathbb{Z}_6$. Then $6|(a - b)$. Hence $3|(a - b)$. so $[a]_3 = [b]_3$ in $\mathbb{Z}_3$, so $\phi([a]_6) = \phi([b]_6)$. Hence ϕ is well-defined. The kernel of ϕ is $\{[0]_6, [3]_6\}$.

(b) Consider $[0]_3$ and $[3]_3$. We have $[0]_3 = [3]_3$ in $\mathbb{Z}_3$ but $[0]_6 \neq [3]_6$ in $\mathbb{Z}_6$. Hence $\phi([0]_3) \neq \phi([3]_3)$, so ϕ is not well-defined.

(c) We claim that $\phi\colon \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi([a]_n) = [a]_m$ is well-defined iff $m|n$. First suppose that $m|n$. If $[a]_n = [b]_n$ in $\mathbb{Z}_n$, then $n|(a-b)$. Since $m|n$, we then have $m|(a-b)$, so $[a]_m = [b]_m$ in $\mathbb{Z}_m$. Hence $\phi([a]_n) = \phi([b]_n)$. Therefore ϕ is well-defined. Now suppose that m does not divide n. Consider $[n]_n$ and $[0]_n$ in $\mathbb{Z}_n$. Clearly $[n]_n = [0]_n$ in $\mathbb{Z}_n$. However, if $[n]_m = [0]_m$ in $\mathbb{Z}_m$, then $m|n-0$, a contradiction. Hence $[n]_m \neq [0]_m$ in $\mathbb{Z}_m$ so ϕ is not well-defined.

(7) (5 pts) Recall that the *dihedral group* of $2n$ elements, denoted by D_n, is the group generated by two elements ρ and σ satisfying the relations $\rho^n = \sigma^2 = e$ and $\sigma\rho = \rho^{-1}\sigma$. Thus the $2n$ elements of D_n are

$$D_n = \{e, \rho, \rho^2, ..., \rho^{n-1}, \rho\sigma, \rho^2\sigma, ..., \rho^{n-1}\sigma\}.$$

This group is the symmetry group of a regular n-gon (so D_4 is the group of symmetries of a square). Prove that A_4 and D_6 are not isomorphic. (Hint: Look at elements of order 6.)

Solution: There is one element of order 6 in D_6, namely ρ, and there are no elements of order 6 in A_4 .

(8) (5 pts) List all the (isomorphism class representatives of) abelian groups of order 240.

Solution: Note that $240 = 2^4 \cdot 3 \cdot 5$. The abelian groups of order 240 are $\mathbb{Z}_{2^4} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(9) (5 pts) Let G be a group and recall that $Z(G)$ is the center of G (look up the definition if you don't remember). Also recall that $Z(G)$ is a normal subgroup of G, so that we may consider the quotient $G/Z(G)$. Prove that, if $G/Z(G)$ is cyclic, then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $Z(G)x$, show that every element of G can be written in the form $x^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.)

Solution: Let $g \in G$ and consider $Z(G)g \in G/Z(G)$. Since $G/Z(G)$ is cyclic, it is generated by some $Z(G)x$, so there is $k \in \mathbb{Z}$ such that $Z(G)g = (Z(G)x)^k = Z(G)x^k$. So $g = z_1x^k$ for some $z_1 \in Z(G)$. Similarly if $h \in G$, then $h = z_2x^j$ for some $z_2 \in Z(G)$ and $j \in \mathbb{Z}$. Therefore $gh = z_1x^kz_2x^j = z_1z_2x^{j+k} = z_2x^kz_1x^j = hg$, so G is abelian.