Math 306 Topics in Algebra, Spring 2013
Homework 10 Solutions

(1) (5 pts) Show that the sequence

$$C = \left(\mathbb{Z}/8\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{d_1} \mathbb{Z}/4\mathbb{Z}\right),$$

where each d_i is multiplication by 4, is a chain complex and compute its homology (where that makes sense).

Solution: The question only makes sense where there are two maps, and incoming and an outgoing one, so we check the chain complex condition and compute the homology at the inner two modules. At $\mathbb{Z}/4\mathbb{Z}$, $\operatorname{im} d_3 = \{0\} \subset \ker d_2 = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z}$, and at $\mathbb{Z}/8\mathbb{Z}$, $\operatorname{im} d_2 = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z} \subset \ker d_1 = \mathbb{Z}/8\mathbb{Z}$. This is therefore a chain complex.

For homology, we get

$$\ker d_2 / \operatorname{im} d_3 \cong (\mathbb{Z}/2\mathbb{Z})/\{0\} \cong \mathbb{Z}/2\mathbb{Z}, \qquad \ker d_1 / \operatorname{im} d_2 \cong (\mathbb{Z}/8\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$$

(2) (7 pts) Prove the *Splitting Lemma*: For a short exact sequence of modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the following are equivalent:

(a) There exists a map $f' \colon B \to A$ such that $f' \circ f$ is the identity on A.
(b) There exists a map $g' \colon C \to B$ such that $g \circ g'$ is the identity on C.
(c) $B \cong A \oplus C$ (so this short exact sequence can be replaced by the short exact sequence of the form $0 \to A \to A \oplus C \to C \to 0$).

A sequence that satisfies these conditions is called *split exact*.

Solution: To show that (c) implies both (a) and (b), let f' be the natural projection of the direct sum onto A, and take as g' the natural injection of C into the direct sum.

To prove that (a) implies (c), first note that any member of B is in the set $\ker f' + \operatorname{im} f$. This follows since for all b in B, $b = (b - f(f'(b))) + f(f'(b))$; $f(f'(b))$ is obviously in $\operatorname{im} f$, and $b - f(f'(b))$ is in $\ker f'$ since

$$f'(b - f(f'(b))) = f'(b) - f'(f(f'(b))) = f'(b) - (f' \circ f)(f'(b)) = f'(b) - f(b) = 0.$$

Next, the intersection of $\operatorname{im} f$ and $\ker f'$ is 0, since if there exists an a in A such that $f(a) = b$, and $f'(b) = 0$, then $0 = f'(f(a)) = a$ and therefore $b = 0$. This proves that B is the direct sum of $\operatorname{im} f'$ and $\ker f$. So for all b in B, b can be uniquely written as $b = f(a) + k$ for some a in A and k in $\ker f'$.

By exactness, g is onto and so for any c in C there exists some $b = f(a) + k$ such that

$$c = g(b) = g(f(a) + k) = g(k).$$

Therefore for any c in C, there exists a k in $\ker f'$ such that $c = g(k)$. We also know that $g(\ker f') = C$.

If $g(k) = 0$, then k is in $\operatorname{im} f$; since the intersection of $\operatorname{im} f$ and $\ker f' = 0$, $k = 0$. Therefore the restriction of the homomorphism g to $\ker f'$ is an isomorphism and so $\ker f'$ is isomorphic to C.

Finally, $\operatorname{im} f$ is isomorphic to A due to the exactness at A, and so B is isomorphic to the direct sum of A and C, which proves (3).

To show that (b) implies (c), use a similar argument. Any member of B is in the set $\ker g + \operatorname{im} g'$; since for all b in B, $b = (b - g'(g(b))) + g'(g(b))$, which is in $\ker g + \operatorname{im} g'$. The intersection of $\ker g$ and $\operatorname{im} g'$ is 0, since if $g(b) = 0$ and $g'(c) = b$, then $0 = g(g'((c)) = c$.

By exactness, $\operatorname{im} f = \ker g$, and since f is an injection, $\operatorname{im} f$ is isomorphic to A, so A is isomorphic to $\ker g$. Since $g \circ g'$ is a bijection, g' is an injection, and thus $\operatorname{im} g'$ is isomorphic to C. So B is again the direct sum of A and C.

(3) (5 pts) Recall the *Short Five Lemma* from lecture: Suppose the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

of modules commutes and the rows are exact. Then
(a) α, γ injections $\Longrightarrow \beta$ is an injection;
(b) α, γ surjections $\Longrightarrow \beta$ is a surjection;
(c) α, γ isomorphism $\Longrightarrow \beta$ is an isomorphism;
Part (a) was proved in lecture. Prove part (b). (Note that part (c) follows from (a) and (b) immediately.)

Solution: Let b' be an element in B'. The image of b' in C' has a preimage in C, call it c. Since B maps onto C by exactness, let b be an element of B that maps to c.

By commutativity, the image of b in C' is the same regardless of how we go around the square. Thus $b' - \beta(b)$ must map to 0 in C'. By exactness, $b' - \beta(b)$ has some preimage a' in A'. Since α is surjective, let $a' = \alpha(a)$ for some a in A.

Finally consider $b +$ (the image of a in B). Apply β, and b becomes $b' - (b' - \beta(b)) = \beta(b)$, while the image of a is $b' - \beta(b)$. The result is b', and β is surjective.

(4) (5 pts) Section 10.5, problem 3 (p. 403)

Solution: If $P_1 \oplus P_2$ is free, then it is a summand of a free module, i.e. $P_1 \oplus P_2 \oplus M$ is free for some module M. But then P_1 and P_2 are also evidently summands of a free module.

On the other hand, if P_1 and P_2 are free, then $P_1 \oplus M$ and $P_2 \oplus N$ are also free for some modules M and N. But since a direct sum of free modules is free (easy to see), $P_1 \oplus M \oplus P_2 \oplus N \cong P_1 \oplus P_2 \oplus M \oplus N$ is thus free. It follows that $P_1 \oplus P_2$ is then a summand of a free module and is thus projective.

(5) (5 pts) Section 10.5, problem 7(a) (p. 404)

Solution: If A is projective, then there exists a module B and a free abelian group F (i.e. a free module over $\mathbb{Z}$ such that $A \oplus B \cong F \cong \mathbb{Z}^n$, where n is the number of generators of the free abelian group F. But $(a, 0)$ has finite order in $A \oplus B$ for any a, whereas no element of $\mathbb{Z}^n$ has finite order. (A quicker way to say this is that any $\mathbb{Z}$-module is torsion-free, since $\mathbb{Z}$ is an integral domain. On the other hand, a finite group is a torsion module.)

(6) (5 pts) Section 10.5, problem 14(a) (p. 404)

Solution: For one direction, we assume the sequence

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

is split exact. To show that the associated Hom sequence is exact, it suffices to show that ψ' is surjective. To that end, let $\alpha \colon L \to D$ be a module homomorphism. By an earlier problem, since the original sequence is split, we know ϕ has an inverse, call it λ. Then $\alpha \circ \lambda \colon M \to D$ is a module homomorphism and

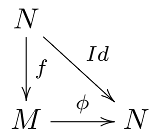
$$\psi'(\alpha \circ \lambda) = \alpha \circ \lambda \circ \phi = \alpha \circ Id = \alpha.$$

Thus ψ' is surjective, and the Hom sequence is exact.

Conversely, suppose the Hom sequence is exact for all D. In particular, setting $D = N$, we have that the map

$$\mathrm{Hom}_R(D, M) \xrightarrow{\psi'} \mathrm{Hom}_R(N, N)$$

is surjective. Then $Id \in \mathrm{Hom}_R(N, N)$ has a lift, i.e. there exists $f \in \mathrm{Hom}_R(D, M)$ such that the diagram



commutes. In other words, ϕ has an inverse, and so the sequence splits by an earlier problem.

(7) (5 pts) Section 10.4, problem 2 (p. 375)

Solution: To see that $2 \otimes 1$ is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, observe that $2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2$. The first equality is true since 1 is an element of $\mathbb{Z}$ and the second is true by the relations in the tensor product. But $1 \otimes 2 = 1 \otimes 0 = 0 \otimes 0$ since $2 = 0 \in \mathbb{Z}/2\mathbb{Z}$.

On the other hand, we cannot do this in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, since $1 \notin 2\mathbb{Z}$. There is still a possiblity that one of the other two relations we mod out by in the tensor product produces $0 \otimes$. One of them is $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, but this does not work since in $\mathbb{Z}_2$, 1 only decomposes as $1 \oplus 0$, and that just yields the unhelpful equation $2 \otimes 0 = 2 \otimes 0$. The other relation is $(m_1 + m_2) \oplus n = m_1 \oplus n + m_2 \oplus n$, in which case each of m_1 and m_2 has to have one factor of 2 to be moved over to the right side (and thus make it zero), but what is left also must be in $2\mathbb{Z}$ so it has to have another factor of 2. Thus $m_1 = 4a$ and $m_2 = 4b$ for some $a, b \in \mathbb{Z}$ and so $m_1 + m_2 = 4(a + b)$. On the other hand, we must have $m_1 + m + 2 = 2$ (since we want $(m_1 + m_2) \oplus n = 2 \oplus 1$), which means $2 = 4(a + b)$, but this is impossible since a and b are integers.