Math 306 Topics in Algebra, Spring 2013
Homework 2 Solutions

(1) (5 pts/part)
  (a) Section 1.7, problem 5 (p. 44). (The kernel of an action is defined on page 43.)
  (b) An action of a group G on a set S is said to be *faithful* if for any $g \in G$, $g \neq e$, there exists an $s \in S$ such that $gs \neq s$. Show that an action is faithful if and only if the associated homomomorphism $\phi\colon G \to \mathrm{Sym}(S)$ has trivial kernel.

Solution:
  (a) The homomorphism ϕ is defined as

$$\begin{aligned} \phi\colon G &\longrightarrow \mathrm{Sym}(S) \\ g &\longmapsto \sigma_g \end{aligned}$$

where

$$\begin{aligned} \sigma_g\colon S &\longrightarrow S \\ s &\longmapsto gs \end{aligned}$$

So kernel of ϕ is all g such that σ_g is is the identity map from S to S (i.e. the trivial permutation of S). In other words,

$$\ker \phi = \{g \in G \mid gs = s \text{ for all } s \in S\}.$$

But this is precisely the kernel of the action $\phi\colon G \times S \to S$ as defined in the book.
  (b) The kernel of the action is trivial if there is no g such that $gs = s$ for any s. In other words, for all g, there exists an s such that $gs \neq s$. But this is precisely the definition of a faithful action.

(2) (5 pts) Recall that a group G acts on a set S *transitively* if, for every $s, s' \in S$, there exists a $h \in G$ such that $hs' = s$. Also recall that the *stabilizer* of an element $s \in S$ is the subgroup $G_s = \{g \in G \mid gs = s\}$. Show that if G acts transitively on S, then all stabilizers are conjugate.

Solution: If G_s and $G_{s'}$ are stabilizers of s and s', we want to show that there exists an element $h \in G$ such that $hG_sh^{-1} = G_{s'}$. Let h be an element for which $hs' = s$ (we know such an element exists by transitivity) and let $g \in G_s$. Then $ghs' = gs$, and

$$h^{-1}ghs' = h^{-1}gs = h^{-1}s = s'.$$

So $(h^{-1}gh)s' = s'$. So what we have is that if an element g stabilizes s, a conjugate of it stabilizes s'. Reversing the roles of s and s' and starting with $g \in G_{s'}$ shows the a conjugate every element in the stabilizer of s' stabilizes s, and the desired result follows.

(3) (4 pts/part) The group $G = \langle (123)(45) \rangle$ is of order 6 and it acts on the set $\{1, 2, 3, 4, 5\}$ (by permuting it).
  (a) Determine $\mathrm{Orb}(k)$ for $1 \leq k \leq 5$.
  (b) Determine G_k for $1 \leq k \leq 5$.
  (c) Use parts (a) and (b) to verify that $|\mathrm{Orb}(k)| = |G|/|G_k|$ for $1 \leq k \leq 5$.

Solution:
  (a) $\mathrm{Orb}(1) = \mathrm{Orb}(2) = \mathrm{Orb}(3) = \{1, 2, 3\}$ and $\mathrm{Orb}(4) = \mathrm{Orb}(5) = \{4, 5\}$.
  (b) $G_1 = G_2 = G_3 = \langle (45) \rangle$ and $G_4 = G_5 = \langle (123) \rangle$.
  (c) Straightforward.

(4) This problem is about normalizers, which we used in the proof of the Extended Sylow's Theorem.
  (a) (4 pts) Show that, for any nonempty subset S of a group G, $N_G(S)$ is a subgroup of G.
  (b) (4 pts) Show that, for any $S \subset G$, $Z(G)$ (the center of G) is a subgroup of $N_G(S)$.
  (c) (5 pts) Let $G = S_3$. Show that the normalizer of $\{1, (123), (132)\}$ is all of S_3.

(d) (5 pts) Let S be the set of all subsets of G. There is an action of G on S given by conjugation. Show that, for $T \in S$,

$$G_T = N_G(T).$$

(Here G_T is the stabilizer of $T \in S$.)

Solution:

(a) The normalizer is

$$N_G(S) = \{g \in G \mid gSg^{-1} = S\}.$$

Clearly $e \in N_G(S)$. If $g \in N_G(S)$, then multiplying $gSg^{-1} = S$ from the left by g^{-1} and the right by g gives $S = g^{-1}Sg$, so $g^{-1} \in N_G(S)$. If $g_1, g_2 \in N_G(S)$, then

$$(g_1g_2)S(g_1g_2)^{-1} = (g_1g_2)S(g_2^{-1}g_1^{-1}) = g_1(g_2Sg_2^{-1})g_1^{-1} = g_1Sg_1^{-1} = S,$$

so $g_1g_2 \in N_G(S)$.

(b) If $g \in Z(G)$, then $gs = sg$ or $gsg^{-1} = s$ for all $s \in G$ and in particular for all $s \in S$, so $Z(G) \subset N_G(S)$.

(c) We have to conjugating this set by the elements of S_3 and show that the result is always the same set. This is straightforward.

(d) By definition, since the action is conjugation,

$$G_T = \{g \in G \mid g(T) = T\} = \{g \in G \mid gTg^{-1} = T\}.$$

But this is precisely the definition of $N_G(T)$.

(5) (4 pts/part) The last part of this problem was used in the example of the classification of groups of order 12. Let H and K be subgroups of a group G. Show the following.

(a) If $H \cap K = \{1\}$, the product map

$$p\colon H \times K \longrightarrow G$$
$$(h,k) \longmapsto hk$$

is injective and the image is the subset HK of G.

(b) If either H or K is a normal subgroup of G, then the product sets HK and KH are equal, and HK is a subgroup of G.

(c) If H and K are normal, $H \cap K = \{1\}$, and $HK = G$ (as a set), then G is isomorphic to the product group $H \times K$.

Solution:

(a) Let (h_1, k_1), (h_2, k_2) be elements of $H \times K$ such that $h_1k_1 = h_2k_2$. Multiplying both sides on the left by h_1^{-1} and on the right by k_2^{-1}, we get $k_1k_2^{-1} = h_1^{-1}h_2$. Since $H \cap K = \{1\}$, $k_1k_2^{-1} = h_1^{-1}h_2 = 1$, and so $h_1 = h_2$ and $k_1 = k_2$.

(b) Suppose H is a normal subgroup of G, and let $h \in H$ and $k \in K$. Note that $kh = (khk^{-1})k$. Since H is normal, $khk^{-1} \in H$, and so $kh \in HK$, so $KH \subset HK$. The proof of the other inclusion is similar.
To show HK is a subgroup, we have: For closure under multiplication, note that in a product $(hk)(h'k') = h(kh')k'$, the middle term kh' is in $KH = HK$, say $kh' = h''k''$. Then $hkh'k' = (hh'')(k''k') \in HK$. For closure under inverses, we have $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. And finally $1 = 1 \cdot 1 \in HK$.
The proof for when K is normal is same; just switch roles of H and K.

(c) Assume that both subgroups are normal and that $H \cap K = \{1\}$. Consider the product $(hkh^{-1})k^{-1} = h(kh^{-1}k^{-1})$. Since K is a normal subgroup, the left side is in K. Since H is normal, the right side is in H. Thus this product is in the intersection so it has to be 1. Thus $hkh^{-1}k^{-1} = 1$ or $hk = kh$. The fact that p is a homomorphism now follows directly: In the group $H \times K$, the product is $(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)$, and this element corresponds to $h_1h_2k_1k_2$ in G, while in G the products h_1k_1 and h_2k_2 multiply as $h_1k_1h_2k_2$. Since $h_2k_1 = k_1h_2$, the products are equal. Part (a) shows that p is injective, and the assumption that $HK = G$ shows that p is surjective.

(6) (3 pts) What are the orders of the Sylow p-subgroups of a group of order 700?

Solution: 2^2, 5^2, and 7.