Math 306 Topics in Algebra, Spring 2013
Homework 3 Solutions

(1) (5 pts) Section 4.5, problem 4 (p. 146), but only for D_6 (which the book calls D_{12}).

Solution: Need to find one Sylow p-subgroup and conjugate it, for $p = 2$ and $p = 3$. It turns out that there are three Sylow 2-subgroups: $\{e, \sigma, \rho^3, \sigma\rho^3\}$, $\{e, \rho^3, \sigma\rho^2, \sigma\rho^5\}$, and $\{e, \sigma\rho, \rho^3, \sigma\rho^4\}$; and one Sylow 3-subgroup: $\{e, \rho^2, \rho^4\}$.

(2) (5 pts) Use Extended Sylow's Theorem to verify that, if $|G| = 24$, then $n_2 = 1$ or 3 and $n_3 = 1$ or 4. Then find n_2 and n_3 for S_4 by finding all Sylow 2-subgroups and all Sylow 3-subgroups of S_4.

Solution: Since $|G| = 24 = 2^3 \cdot 3$, we have

$$n_2 \equiv 1 \pmod 2 \text{ and } n_2|3 \Longrightarrow n_2 = 1 \text{ or } 3;$$
$$n_3 \equiv 1 \pmod 3 \text{ and } n_3|8 \Longrightarrow n_3 = 1 \text{ or } 4.$$

For S_4, we have that $n_2 = 3$ and $n_3 = 4$ since more than one element of S_4 has order 2 or 3 and there are thus more than one subgroup of orders 2 or 3.

(3) (5 pts/part)
- (a) Prove that there are no simple groups of order 20.
- (b) Prove that a group of order 175 must have normal subgroups of orders 7 and 25.

Solution:
- (a) Let $|G| = 20$. Then $n_5 \equiv 1 \pmod 5$ and $n_5|4$, so $n_5 = 1$. Therefore the Sylow 5-subgroup must be normal, and G is not simple.
- (b) Let $|G| = 175$. Then $n_5 \equiv 1 \pmod 5$ and $n_5|7$, so $n_5 = 1$. So there is only one Sylow 25-subgroup. Similarly there is only one Sylow 7-subgroup and both therefore have to be normal.

(4) (7 pts) Classify groups of order 21.

Solution: Let G be a group of order $21 = 3 \cdot 7$. Sylow's Theorem show that the Sylow 7-subgroup K must be normal. But the possibility that there are 7 conjugate Sylow 3-subgroups is not ruled out, and this case indeed arises.

Let x denote a generator for K and y a generator for one of the Sylow 3-subgroups H. Then $x^7 = 1$, $y^3 = 1$, and , since K is normal, $yxy^{-1} = x^i$ for some $i < 7$. We can restrict the possible exponents by using the relation $y^3 = 1$. Namely, this relation implies that

$$x = y^3xy^{-3} = y^2x^iy^{-2} = yx^{i^2}y^{-1} = x^{i^3}.$$

Hence $i^3 \equiv 1 \pmod 7$. This means that $i = 1, 2$ or 4.

Case 1: $yxy^{-1} = x$. The group is abelian, and by the classification theorem for finite abelian groups, it is

$$\mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{21}.$$

Case 2: $yxy^{-1} = x^2$. These is no familiar description for this group. In terms of generators and relations, it is thus given by

$$G = \langle x, y \mid x^7 = 1, y^3 = 1, yx = x^2y \rangle.$$

Case 3: $yxy^{-1} = x^4$. Replace y by y^2, which also generates H, to reduce to the previous case:

$$y^2xy^{-2} = yx^4y^{-1} = x^{16} = x^2.$$

(5) (5 pts) Recall that, in the proof of the classification of groups of order 12, we let H be a Sylow 2-subgroup of G and K be a Sylow 3-subgroup of G. Prove that at least one of these subgroups is normal.

Solution: Suppose K is not normal. Then there has to be more than one Sylow 3-subgroup, and this number has to be four. Denote these four distinct conjugate Sylow 3-subgroups by $K = K_1, K_2, K_3, K_4$. Since $|K_i| = 3$, the intersection of any two of these groups has to be identity (if it were not and they had a common non-identity element, this element would be of order 3 necessarily and hence generate both groups so the groups would not be distinct). The total number of elements between the K_i is thus 9 (they all have the identity in common). So that leaves 3 non-identity elements, which, together with the identity, comprise a Sylow 3-subgroup (there can't be any intersection between Sylow 2-subgroups and Sylow 2-subgroups, again because of the incompatible orders of elements). This accounts for all the elements of G and there hence cannot be another Sylow 2-subgroup. So the single Sylow 2-subgroup is normal.

(6) (7 pts) Section 4.5, problem 13 (p. 147).

Solution: The argument here is essentially the same as in the previous problem. Let G be a group of order $56 = 23 \cdot 7$. Let n_2 and n_7 denote the number of Sylow 2-subgroups and the number of Sylow 7-subgroups of G, respectively. Sylow's Theorem tells us that $n_7 \equiv 1 \pmod 7$ and that n_7 divides 8. Hence there are two possibilities: either $n_7 = 1$ or $n_7 = 8$.

If $n_7 = 1$, then there is a unique Sylow 7-subgroup and this is a normal subgroup of order 7. In this case, G is not simple.

Suppose then that $n_7 = 8$. Let $S_1, S_2, ..., S_8$ denote the eight Sylow 7-subgroups. Note that each S_i contains the identity element and six elements of order 7 (since $S_i \cong Z_7$). Now if $i \neq j$, the intersection $S_i \cap S_j$ is a proper subgroup of both S_i and S_j, and Lagrange's Theorem tells us that $S_i \cap S_j = 1$. Hence each S_i contains six elements of order 7 that lie in no other Sylow 7-subgroup of G and we conclude that between them the eight Sylow 7-subgroups contain $8 \cdot 6 = 48$ elements of order 7. There are only 8 remaining elements in G which do not have order 7 (including the identity element in these eight) and any Sylow 2-subgroup must therefore consist of some of these eight elements not of order 7. However, a Sylow 2-subgroup of G has order $2^3 = 8$ and we therefore conclude that there is exactly one Sylow 2-subgroup of G consisting of these eight elements. Hence, if $n_7 = 8$, then $n_2 = 1$ and G has a normal subgroup of order 8.

Thus in either case, G is not simple.