Math 306 Topics in Algebra, Spring 2013
Homework 4 Solutions

(1) (4 pts) Show that any abelian group can be made into a ring without unit by declaring the product of any two elements to be the identity in G.

Solution: Straightforward verification of ring axioms.

(2) (5 pts) Show that $\mathbb{Z}_n$ is a field if and only if n is a prime (in which case we denote it by $\mathbb{F}_n$).

Solution: See Example (2) on page 226.

(3) (5 pts) Show that $GL_n(F)$, the set of all $n \times n$ matrices with non-zero determinant (where the determinant is computed the same way as in the case $F = \mathbb{R}$), is a group. You may use facts about determinant you know from linear algebra.

Solution: See bottom of page 34.

(4) Let $\mathcal{M}_{m\times n}$ be the additive group of all $m \times n$ matrices with coefficients in a field $\mathbb{F}$.
  (a) (5 pts) Show $\mathcal{M}_{m\times n}$ is a vector space over $\mathbb{F}$.
  (b) (4 pts) Find a basis and dimension of this space.
  (c) (5 pts) Show $\mathcal{M}_{m\times n}$ is isomorphic to $\mathbb{F}^{mn}$.

Solution:
  (a) Straightforward verification of the axioms.
  (b) The basis can be taken to be the matrices with 1 in one slot and zero in all the others. This is thus an mn-dimensional space over $\mathbb{F}$.
  (c) Send a matrix to the vector obtained by reading off the entries of the matrix in some way.

(5) (5 pts) Suppose B and B' are two bases for an n-dimensional vector space V and suppose $f\colon V \to V$ is an endomorphism. Let $T_{B,B}$ and $T_{B',B'}$ be matrix representations of f with respect to the two bases. Show that $T_{B,B}$ and $T_{B',B'}$ are conjugate, i.e. that there exists an $n \times n$ matrix A such that $AT_{B,B}A^{-1} = T_{B',B'}$.

Solution: Let $a\colon V \to V$ be the change-of-basis isomorphism that sends the basis B to basis B' and let its matrix representation be A. Let f_B and $f_{B'}$ be f, but with respect to the two different bases (so they're really the same map, but the notation just remembers what the basis is). Then $f_{B'} = a \circ f_B \circ a^{-1}$, and the matrix representation of this equation is precisely $AT_{B,B}A^{-1} = T_{B',B'}$ (since matrix multiplication is defined to represent compositions of linear transformations).

(6) (4 pts/part)
  (a) Show that $\mathbb{F}$ is a one-dimensional vector space over itself.
  (b) Show that $GL(\mathbb{F}) \cong \mathbb{F}^\times$.

Solution:
  (a) Straightforward verification of vector space axioms. A basis is $\{1\}$ (in fact, any non-zero element will do).
  (b) It is not hard to show that the map $GL(\mathbb{F}) \to \mathbb{F}^\times$ given by $f \mapsto f(1)$ is an isomorphism. (This can also be thought of as: $GL(\mathbb{F})$ is isomorphic to the (multiplicative) group of invertible 1×1 matrices over $\mathbb{F}$, which is $F^\times$).

(7) (a) (3 pts) Show that, if $T \in \mathrm{End}(V)$ fixes vectors $v_1, ..., v_n$, i.e. $T(v_i) = v_i$, then it fixes the subspace generated by those vectors. In particular, $\mathrm{Span}\{v_1, ..., v_n\}$ is an invariant subspace for T.
  (b) (3 pts) Suppose $T \in \mathrm{End}(V)$. Show that the kernel of T (i.e. the nullspace of T) and the range of T are invariant subspaces of T.

Solution:

(a)
$$T(\sum a_i v_i) = a_i \sum T(v_i) = a_i v_i.$$

(b) If $v \in \ker(T)$, then $T(v) = 0$, but since $0 \in \ker(T)$, it follows that $T(v) \in \ker(T)$. For range, by definition we have $T(\mathrm{range}(T)) \subset \mathrm{range}(T)$.

(8) (5 pts) Suppose $T \in \mathrm{End}(V)$. Let λ be an eigenvalue of T and let V_λ be the eigenspace corresponding to λ (eigenvectors and eigenvalues are defined the same way for matrices over general field $\mathbb{F}$ as over $\mathbb{R}$). Show that V_λ is a T-invariant subspace of V.

Solution: By definition, $V_\lambda = \{v \in V \mid Tv = \lambda v\}$. Thus, for $v \in V_\lambda$,
$$T(Tv) = T(\lambda V) = \lambda T v = \lambda(Tv),$$
and hence $Tv \in V_\lambda$.