Math 306 Topics in Algebra, Spring 2013
Homework 5 Solutions

Note: In the problems where you are asked to verify that something is a representation, don't forget to say why the image of ϕ is indeed in $GL(V)$, i.e. why ϕ_g is an automorphism of V for all $g \in G$. Usually a quick explanation will suffice.

(1) (5 pts) Prove that, for $A, B \in GL_n(F)$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Deduce that, if A and B are conjugate, they have the same trace.

Solution:

$$\begin{aligned}\mathrm{Tr}(AB) = \sum_{k=1}^{n}(AB)_{kk} &= \sum_{k=1}^{n}\left(\sum_{l=1}^{n} a_{kl}b_{lk}\right) \\ &= \sum_{l=1}^{n}\left(\sum_{k=1}^{n} b_{lk}a_{kl}\right) \\ &= \sum_{l=1}^{n}(BA)_{ll} = \mathrm{Tr}(BA)\end{aligned}$$

If A and B are conjugate, then we have, for some $C \in GL_n(F)$,

$$\mathrm{Tr}(A) = \mathrm{Tr}(CBC^{-1}) = \mathrm{Tr}(CC^{-1}B) = \mathrm{Tr}(B).$$

(2) Let W_1 and W_2 be subspaces of a vector space V.

(a) (3 pts) Show that W_1 and W_2 are independent if an only if $W_1 \cap W_2 = \{0\}$.

(b) (5 pts) Show that there is an isomorphism $W_1 \times W_2 \cong W_1 \oplus W_2$.

Solution:

(a) W_1 and W_2 are independent if and only if, for all $w_1 \in W_1$ and $w_2 \in W_2$, $w_1 = -w_2$ implies that $w_1 = w_2 = 0$. Since $w_2 \in W_2$ iff $-w_2 \in W_2$, we can replace $-w_2$ by w_2 in the above and say that W_1 and W_2 are independent if and only if, for all $w_1 \in W_1$ and $w_2 \in W_2$, $w_1 = w_2$ implies that $w_1 = w_2 = 0$. In other words, $W_1 \cap W_2 = \{0\}$.

(b) Define a map ϕ by

$$\begin{aligned}\phi\colon W_1 \times W_2 &\longrightarrow W_1 \oplus W_2 \\ (w_1, w_2) &\longmapsto w_1 + w_2\end{aligned}$$

This is a linear transformation since

$$\begin{aligned}\phi(c(w_1, w_2) + c'(w_1', w_2')) &= \phi((cw_1, cw_2) + (c'w_1', c'w_2')) \\ &= (cw_1 + cw_2) + (c'w_1' + c'w_2') \\ &= c(w_1 + w_2) + c'(w_1' + w_2') \\ &= c\phi((w_1, w_2)) + c'\phi((w_1', w_2')).\end{aligned}$$

To show it is injective, suppose $\phi((w_1, w_2)) = \phi((w_1', w_2'))$ so $w_1 + w_2 = w_1' + w_2'$, and hence $w_1 - w_1' = w_2 - w_2'$. The left side is a vector in W_1 and the right is a vector in W_2. But since these two subspaces are independent (otherwise we would not talk about their direct sum), by the previous part if follows that $w_1 - w_1' = w_2 - w_2' = 0$. In other words, $w_1 = w_1'$ and $w_2 = w_2'$, and so $(w_1, w_2) = (w_1', w_2')$.

For surjectivity, any element of $W_1 \oplus W_2$ looks like $w_1 + w_2$ and is the image of (w_1, w_2).

(3) (a) (2 pts) Show that $\phi\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}^\times$ given by $\phi(m) = i^m$ is a (one-dimensional) representation.

(b) (2 pts) More generally, show that $\phi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ given by $\phi(m) = e^{2\pi i m/n}$ is a (one-dimensional) representation.

(c) (3 pts) Why does part (b) generalize part (a)?

Solution:

(a) $\phi(m+m') = i^{m+m'} = i^m i^{m'} = \phi(m)\phi(m')$, and, since $i^4 = 1$, taking powers of i modulo 4 is compatible with addition modulo 4 in $\mathbb{Z}/4\mathbb{Z}$, so the map is well-defined. (It's also important to note that, for one-dimensional representations, multiplication is the group operation in $F^\times$. In general it is composition, but this reduces to multiplication when $V = F$.) In this part and then next, ϕ_m is an isomorphism since it is an invertible element of F.

(b) $\phi(m+m') = e^{2\pi i(m+m')/n} = e^{2\pi i m/n} \cdot e^{2\pi i m'/n} = \phi(m)\phi(m')$, and $(e^{2\pi i m/n})^n = 1$, so again we have compatibility of operations. This is a well-defined map since $e^{2\pi i n/n} = e^{2\pi i} = 1$.

(c) When $n = 4$, we have $e^{2\pi i m/4} = \left(e^{2\pi i/4}\right)^m = (\cos 2\pi/4 + i \sin 2\pi/4)^m = (\cos\pi/2 + i\sin\pi/2)^m = i^m$.

(4) (5 pts) Let $V = \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$. Show that, for each $1 \le m \le n-1$, there is a representation $\mathbb{Z}/n\mathbb{Z} \to GL(V)$ given by

$$1 \longmapsto \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}$$

Solution: Since we are given what ϕ does on a generator, it follows that, for $k \in \mathbb{Z}/n\mathbb{Z}$, the map is given by

$$k \longmapsto \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}^k$$

(addition goes to matrix product). This is clearly a homomorphism since

$$k_1 + k_2 \longmapsto \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}^{k_1+k_2} = \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}^{k_1} \cdot \begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}^{k_2}$$

However, we need to see why this is well-defined, i.e. compatible with the relation that $n = 0$ in $\mathbb{Z}/n\mathbb{Z}$. In other words, we need the nth power of the matrix the generator maps to to be the identity matrix in $GL(\mathbb{R}^2)$. But this is true since (as can be shown by induction),

$$\begin{pmatrix} \cos\frac{2\pi m}{n} & -\sin\frac{2\pi m}{n} \\ \sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{pmatrix}^k = \begin{pmatrix} \cos\frac{2k\pi m}{n} & -\sin\frac{2k\pi m}{n} \\ \sin\frac{2k\pi m}{n} & \cos\frac{2k\pi m}{n} \end{pmatrix}.$$

When $k = n$, this gives the identity matrix.

To see that this map really lands where it is supposed to, since the determinant of the matrix that the generator maps to is 1, and since determinant is multiplicative, it follows that the determinant of any power of that matrix is also 1. So for every k, the image of k is an invertible matrix and hence it represents an automorphism of $\mathbb{R}^2$.

(5) This problem studies the *regular representation* of a group. Let G be a finite group or order n and F a field. Define $F[G]$ be the n-dimensional vector space generated by the elements of G. In other words, $F[G]$ is a vector space whose basis is G and whose elements are formal linear combinations of elements of G with coefficients in F. Now define the *left regular representation of* G by

$$\begin{aligned} L\colon G &\longrightarrow GL(F[G]) \\ g &\longmapsto L_g(v) = gv \end{aligned}$$

The multiplication here means the following: Since $v = \sum a_i g_i$, let $gv = g\sum a_i g_i = \sum a_i(gg_i)$. Similarly define the *right* regular representation $R\colon G \to GL(F[G])$ by $R_g(v) = vg^{-1}$.

(a) (5 pts) Show that L is a representation, and then briefly say what changes for R.

(b) (3 pts) Recall that representations correspond to linear actions. What actions to R and L correspond to?

Solution:

(a) That L is a homomorphism follows from the fact that G is a group. Namely, we have

$$g_1 g_2 \longmapsto L_{g_1 g_2}(v) = (g_1 g_2)v = g_1(g_2 v) = L_{g_1}(L_{g_2}(v)).$$

To see that L_g is an automorphism of $F[G]$, it suffices to consider what it does on the basis, namely the elements of G. But, for any $g \in G$, multiplication by g is an isomorphism, so $gG = G$ (as a set). In other words, L_g simply permutes the basis vectors of $F[G]$, and the resulting transformation is thus an isomorphism. The arguments are similar for R.

(b) They correspond to actions $G \times V \to V$ given by $(g, v) \mapsto gv$ and $(g, v) \mapsto vg^{-1}$.