Math 306 Topics in Algebra, Spring 2013
Homework 6 Solutions

(1) (5 pts) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternions (remind yourselves of the relations in this group). Construct a 2-dimensional representation of Q_8 over $\mathbb{C}$ (and show that it is really a representation).

Solution: For example, can set

$$i \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that nothing else needs to be specified. Namely, the relations in Q_8 are

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2,$$

so once a homomorphism is specified on two of the elements i, j, k, it is determined on the others. What should still be checked that this is a homomorphism, i.e. that all the relations in Q_8 are satisfied, and this is straightforward.

(2) (4 pts) Restate the definition of the equivalence of representations in terms of group actions.

Solution: Suppose representations $\phi\colon G \to GL(V)$ and $\phi'\colon G \to GL(V')$ are equivalent, so there is an isomorphism $T\colon V \to V'$ such that, for all $g \in G$, $\phi'_g = T\phi_g T^{-1}$, or $\phi'_g T = T\phi_g$. Representations ϕ and ϕ' determine actions $G \times V \to V$ and $G \times V' \to V'$. The condition that $\phi \sim \phi'$ is then precisely equivalent to the condition that there exists an isomorphism $T\colon V \to V'$ such that, for all $g \in G$, $gT(v) = T(gv)$ (here $T(v)$ is a vector in V' and gv is a vector in V).

(3) (5 pts/part)

(a) Suppose representation $\phi\colon G \to GL(V)$ is decomposable, i.e. there exist nontrivial subspaces W_1, W_2 of V that are G-invariant and $V = W_1 \oplus W_2$. Show that a representation $\phi\colon G \to GL(V)$ is equivalent to the representation $\phi|_{W_1} \oplus \phi|_{W_2}$.

(b) Suppose G is generated by elements $g_1, ..., g_k$ and suppose $\phi\colon G \to GL(V)$ is a representation. Show that, if $\phi_{g_i} = \phi'_{g_i}|_{W_1} \oplus \phi''_{g_i}|_{W_2}$ for some subspaces W_1, W_2 satisfying $V = W_1 \oplus W_2$, then $\phi_g = \phi'_g|_{W_1} \oplus \phi''_g|_{W_2}$ for all $g \in G$.

Solution:

(a) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for W_1 and W_2, respectively. We then know that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V. Since W_i is G-invariant, we have $\phi_g(\mathcal{B}_i) \subset W_i$. So in matrix form, $(\phi_g)_{\mathcal{B}}$ (the matrix of ϕ_g in basis $\mathcal{B}$) can be written as

$$(\phi_g)_{\mathcal{B}} = \begin{pmatrix} (\phi_g|_{W_1})_{\mathcal{B}_1} & 0 \\ 0 & (\phi_g|_{W_2})_{\mathcal{B}_2} \end{pmatrix}$$

But this is precisely the matrix for $\phi|_{W_1} \oplus \phi|_{W_2}$.

(b) The matrix for ϕ_{g_i} is block diagonal, and, since the g_i generate G and any g is a product of the powers of the g_i, the matrix for ϕ_g is the product of the powers of the matrices for ϕ_{g_i}. Since products of block diagonal matrices are block diagonal (not hard to see; it is true because multiplication of block matrices can be carried out by treating each block as a single entry and then multiplying the matrices within each block), each ϕ_g is block diagonal and has W_1, W_2 as invariant subspaces and hence decomposes as $\phi'_g|_{W_1} \oplus \phi''_g|_{W_2}$.

(4) (5 pts) Suppose that $\phi\colon G \to GL(V)$ is an n-dimensional representation. Define $\phi^*\colon G \to GL(V)$ by $\phi^*_g = \phi^T_{g^{-1}}$, where T means the transpose matrix. Show that ϕ^* also defines an n-dimensional representation.

Solution: To simplify notation, we will write $\phi(g)$ instead of ϕ_g. We then have

$$\phi^*(g_1 g_2) = (\phi((g_1 g_2)^{-1}))^T = (\phi(g_2^{-1} g_1^{-1}))^T = (\phi(g_2^{-1})\phi(g_1^{-1}))^T = (\phi(g_1^{-1})^T \phi(g_2^{-1}))^T = \phi^*(g_1)\phi^*(g_2).$$

Here we have used the properties of the transpose and the fact that ϕ is a homomorphism.

(5) (5 pts) Find four irreducible representations of $\mathbb{Z}_4$ over $\mathbb{C}$ (thought of as a 1-dimensional vector space over itself).

Solution: If x is the generator of $\mathbb{Z}_4$, the 1-dimensional (and hence irreducible) representations are given by $x \mapsto \pm 1, \pm i$.

(6) (5 pts) In class, we had two 2-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$, ϕ and ψ, that we showed were equivalent. Show that neither of them is irreducible.

Solution: We already showed in class that, for ϕ, the subspaces generated by e_1 and e_2 are invariant. For ψ, it is not hard to see that, for example, subspaces generated by vectors $(i, 1)$ and $(-i, 1)$ are invariant.

(7) (a) (4 pts) Show that $\mathbb{Z}_3 = \{e, x, x^2\}$ has a 3-dimensional representation ϕ over F given by

$$\phi_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \phi_{x^2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) (4 pts) Is ϕ equivalent to ϕ^*? Justify your answer. (See above for the definition of ϕ^*.)

(c) (4 pts) Show that the representation from part (a) is not irreducible by finding a 1-dimensional subrepresentation.

(d) (5 pts) Can you find more than one of these 1-dimensional subrepresentations? (Hint: The answer depends on F.)

Solution:

(a) Straightforward; just check that $\phi_{x^2} = \phi_x \cdot \phi_x$ etc.

(b) $\phi_x^* = \phi_{x^{-1}}^T = \phi_{x^2}^T = \phi(x)$, so the representations are the same since they are the same on the generator.

(c) Vector $e_1 + e_2 + e_3 = (1, 1, 1)$ is fixed by all the matrices, and so the subspace W spanned by this vector is $\mathbb{Z}_3$-invariant. Hence ϕ has a subrepresentation obtained by restricting it to W.

(d) Suppose a vector v is preserved by ϕ_x, so $\phi_x v = \lambda v$ for some $\lambda \in F$ (this has to be satisfied if v is a vector in a 1-dimensional invariant subspace). In other words, v is an eigenvector for ϕ_x and λ an eigenvalue. To find eigenvalues of ϕ_x, we solve

$$\det(tI - \phi_x) = \begin{vmatrix} t & -1 & 0 \\ 0 & t & -1 \\ -1 & 0 & t \end{vmatrix} = t^3 - 1 = (t-1)(t^2 + t + 1)$$

So one eigenvalue is 1 and its eigenvector is $(1, 1, 1)$, which we already found. Whether there are other eigenvalue depends on whether $t^2 + t + 1$ factors in F. If $F = \mathbb{R}$, it does not. If $F = \mathbb{C}$, it does and we have

$$t^2 + t + 1 = (t - e^{2\pi i/3})(t - e^{4\pi i/3}).$$

Then ϕ_{x^2} has the same eigenvalues, and so there are two more 1-dimensional subspaces spanned by the eigenvectors corresponding to these to eigenvalues, namely $\langle(1, e^{2\pi i/3}, e^{4\pi i/3})\rangle$ and $\langle(1, e^{4\pi i/3}, e^{2\pi i/3})\rangle$.