Math 306 Topics in Algebra, Spring 2013
Homework 7 Solutions

(1) (5 pts) Let G be a finite group. Show that the function

$$\mathbb{C}[G] \times \mathbb{C}[G] \longrightarrow \mathbb{C}$$
$$(f_1, f_2) \longmapsto \langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

defines an inner product on $\mathbb{C}[G]$.

Solution: We have

$$\begin{aligned}
\langle c_1f_1 + c_2f_2, f_3\rangle &= \frac{1}{|G|}\sum_{g\in G}(c_1f_1 + c_2f_2)(g)\overline{f_3(g)}\\
&= \frac{1}{|G|}\sum_{g\in G}\left(c_1f_1(g)\overline{f_3(g)} + c_2f_2(g)\overline{f_3(g)}\right)\\
&= \frac{1}{|G|}\sum_{g\in G}c_1f_1(g)\overline{f_3(g)} + \frac{1}{|G|}\sum_{g\in G}c_2f_2(g)\overline{f_3(g)}\\
&= c_1\frac{1}{|G|}\sum_{g\in G}f_1(g)\overline{f_3(g)} + c_2\frac{1}{|G|}\sum_{g\in G}f_2(g)\overline{f_3(g)}\\
&= c_1\langle f_1, f_3\rangle + c_2\langle f_2, f_3\rangle
\end{aligned}$$

Also

$$\begin{aligned}
\overline{\langle f_1, f_2\rangle} = &\overline{\frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}}\\
&\frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)\overline{f_2(g)}}\\
&\frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}\,\overline{\overline{f_2(g)}}\\
&\frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}\,f_2(g) = \langle f_2, f_1\rangle
\end{aligned}$$

Lastly, we have

$$\langle f, f\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{f(g)} = \frac{1}{|G|}\sum_{g\in G}|f(g)|^2.$$

This sum is greater than or equal to zero and is zero if and only if $f(g) = 0$ for all $g \in G$.

(2) Instead of taking the trace of ϕ_g to define the character, one might try to do the same by taking the determinant of ϕ_g instead. This problem shows that this is not as useful since such a character would tell us nothing about non-abelian simple groups (and these are important).

For ϕ a representation of a finite group G, define a function

$$\det\phi\colon G \longrightarrow \mathbb{C}^\times$$
$$g \longmapsto (\det\phi)(g) = \det(\phi_g)$$

(a) (4 pts) Show $\det\phi$ is a representatation (and hence it's a character since characters are same as representations for one-dimensional representations).

(b) (5 pts) Show that if G is a non-abelian simple group, then $\det\phi$ is the trivial character.

Solution:

(a) Follows from the multiplicativity of the determinant.

(b) Since $\det\phi$ is a homomorphism, its kernel is a normal subgroup of G. Since G is simple, it must be that $\ker(\det\phi) = \{e\}$ or $\ker(\det\phi) = G$. If the former is true, then $\det\phi$ is injective, and its image is a subgroup of $\mathbb{C}^\times$, which is abelian. This G would be isomorphic to an abelian group, but this cannot by by assumption. So it must be that $\ker(\det\phi) = G$, in which case $\det\phi$ is the trivial homomorphism.

(3) (4 pts) Show that $\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi$.

Solution: The matrix for each $(\phi\oplus\psi)_g$ is a block matrix with blocks ϕ_g and ψ_g. The trace of a matrix like that is the sum of the traces of the blocks. Hence $\chi_\phi + \chi_\psi$.

(4) (5 pts/part)

(a) Suppose A is a matrix over $\mathbb{C}$ of finite order, i.e. $A^n = I$ for some positive integer n. Show that the eigenvalues λ_i of A are the nth roots of unity, namely they satisfy $\lambda_i^n = 1$. (Hint: Use the result that A is diagonalizable, which in turn follows from fact that for a representation of a finite group G, there exists a matrix T such that $T\phi_g T^{-1}$ is diagonal for all $g \in G$. Then look up what diagonalizability has to do with eigenvalues.)

(b) Prove that for an irreducible representation of a finite group G,

$$\chi_\phi(g) = \lambda_1 + \cdots + \lambda_d,$$

where λ_i are the eigenvalues of ϕ_g and d is the dimension of ϕ.

(c) Show that, if a complex number ω is a root of unity, then $\omega^{-1} = \overline{\omega}$.

(d) Prove that $\chi_\phi(g^{-1}) = \overline{\chi_\phi(g)}$.

Solution:

(a) If A is diagonalizable, then it is a basic result of linear algebra that its diagonal entries are its (distinct) eigenvalues λ_i. Denote this diagonal matrix by D. Then

$$D^n = (TAT^{-1})^n = TA^nT^{-1} = TIT^{-1} = TT^{-1} = I$$

But powers of a diagonal matrix are obtained by taking powers of its diagonal entries. Thus it follows that $\lambda_i^n = 1$ as desired.

(b) Since G is finite, ϕ_g has finite order. Hence, by the previous part, there exists a matrix T such that $T\phi_g T^{-1} = D$, where D is diagonal with eigenvalues λ_i as the diagonal entries. Since characters take the same value on similar matrices, we have

$$\chi_\phi(g) = \mathrm{Tr}(\phi_g) = \mathrm{Tr}(T\phi_g T^{-1}) = \mathrm{Tr}(D) = \lambda_1 + \cdots + \lambda_d.$$

(c) A conjugate of a complex number $re^{i\theta}$ is $re^{-i\theta}$. If ω is a root of unity, then it has the form $e^{i\theta/n}$ for some n (but the important thing is that $r = 1$; if $r \neq 1$, then $(re^{i\theta})^n = r^n e^{i\theta n}$, and this number could not have size (modulus) 1). Then

$$\omega\overline{\omega} = e^{i\theta/n}e^{-i\theta/n} = e^0 = 1.$$

So $\overline{\omega}$ is the inverse of ω.

(d) Consider

$$\chi_\phi(g^{-1}) = \mathrm{Tr}(\phi_{g^{-1}}) = \mathrm{Tr}(\phi_g^{-1}).$$

Since trace is same on similar matrices, by part (a) we can replace ϕ_g by the diagonal matrix with diagonal entries the eigenvalues λ_i, and consequently we can replace $\phi_{g^{-1}}$ by the diagonal matrix whose entries are λ_i^{-1} (since to obtain the inverse of a diagonal matrix, you take the inverse of the diagonal entries). Thus

$$\mathrm{Tr}(\phi_{g^{-1}}) = \sum_i \lambda_i^{-1}.$$

Since λ_i are roots of unity, by part (b) we have

$$\mathrm{Tr}(\phi_{g^{-1}}) = \sum_i \overline{\lambda}_i.$$

But this sum is precisely $\mathrm{Tr}(\overline{\phi_g})$ (or rather the trace of the conjugate of the diagonal matrix replacing ϕ_g). I.e. the sum is precisely $\overline{\chi_\phi}$.

(5) (5 pts) Recall that, in the proof of the theorem that a representation ϕ is irreducible iff $\langle \chi_\phi, \chi_\phi \rangle = 1$, we assumed $\phi \sim m_1\phi_1 \oplus \cdots \oplus m_s\phi_s$ and then claimed that

$$\langle \chi_\phi, \chi_\phi \rangle = m_1^2 + \cdots + m_s^2.$$

Show that this equation indeed holds.

Solution: The matrix for ϕ is a block matrix whose blocks are the matrices for $m_i\phi_i$. It is then immediate that

$$\chi_\phi = \mathrm{Tr}(\phi) = \sum_{i=i}^{s} \mathrm{Tr}(m_i\phi_i) = \sum_{i=i}^{s} m_i \,\mathrm{Tr}(\phi_i) = \sum_{i=i}^{s} m_i\chi_{\phi_i}$$

(we can pull out m_i since it appears in each diagonal term of the ith block). So then we have

$$\langle \chi_\phi, \chi_\phi \rangle = \langle \sum_{i=i}^{s} m_i\chi_{\phi_i}, \sum_{j=i}^{s} m_j\chi_{\phi_j} \rangle.$$

However, inner product is linear by definition (and you verified this in an earlier problem for the inner product we're using here), so that the above can be computed by "foiling" the inner product. We thus have

$$\langle \chi_\phi, \chi_\phi \rangle = \sum_{1\le i,j\le n} \langle m_i\chi_{\phi_i}, m_j\chi_{\phi_j} \rangle.$$

By orthogonality relations, the above inner products are only nonzero when $i = j$ in which case they are 1 and we get

$$\langle \chi_\phi, \chi_\phi \rangle = \sum_{1\le i\le n} \langle m_i\chi_{\phi_i}, m_i\chi_{\phi_i} \rangle = \sum_{1\le i\le n} m_i^2 \langle \chi_{\phi_i}, \chi_{\phi_i} \rangle = \sum_{1\le i\le n} m_i^2$$

as desired (because of linearity, we were able to take each m_i out as well).

(6) (a) (4 pts) Show that a finite group G is abelian if and only if it has $|G|$ irreducible representations (over $\mathbb{C}$).
(b) (5 pts) Use part (a) to show $\mathbb{Z}/n\mathbb{Z}$ is abelian. Do this without using that $\mathbb{Z}/n\mathbb{Z}$ has n conjugacy classes.

Solution:

(a) A finite group is abelian iff each element is its own conjugacy class. Thus $|G| = |Cl(G)|$. But we know by a theorem from class that $|Cl(G)|$ is the number of irreducible representations of G.

(b) For $0 \le k \le n-1$, define representations

$$\begin{aligned} \phi_k \colon \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{C}^\times \\ m &\longmapsto e^{2\pi i mk/n} \end{aligned}$$

As we know from class and previous homework, each ϕ_k is a well-defined irreducible representation, and all n of them are distinct. By the previous part, it follows that $\mathbb{Z}/n\mathbb{Z}$ is abelian.

(7) This problem explores the regular representation over $\mathbb{C}$ and the associated character. The formula you will in part in part (d) is an important application of representation theory to group theory.

Recall from an earlier homework that the (left) regular representation of a finite group G is given by

$$\begin{aligned} L \colon G &\longrightarrow GL(F[G]) \\ g &\longmapsto L_g(v) = gv. \end{aligned}$$

(We also defined it in class using the dual vector space of $F[G]$, but for this problem, we'll stick to the original definition above.)

(a) (7 pts) Prove that the character of the regular representation is given by

$$\chi_L \colon G \longrightarrow \mathbb{C}$$
$$g \longmapsto \chi_L(g) = \begin{cases} |G|, & g = 1, \\ 0, & g \neq 1. \end{cases}$$

(b) (5 pts) We observed in class that every representation ϕ of G breaks up as

$$\phi \sim m_1\phi_1 \oplus \cdots \oplus m_s\phi_s,$$

where $\phi_1, ..., \phi_s$ is the complete set of irreducible representations of G (some of the m_i's might be zero). Prove that $\langle \chi_\phi, \chi_{\phi_i} \rangle = m_i$. (Hint: Use an earlier exercise and the linearity of the inner product.)

(c) (7 pts) Suppose d_i are the degrees of the irreducible representations ϕ_i of G. Show that if $\phi = L_g$, the m_i's from the previous part are precisely the degrees d_i. In other words, show that the decomposition

$$L \sim d_1\phi_1 \oplus \cdots \oplus d_s\phi_s$$

holds.

(d) (4 pts) Prove that

$$|G| = d_1^2 + \cdots + d_s^2.$$

Solution:

(a) Let $G = \{g_1, ..., g_n\}$, where $n = |G|$. Let $v = g_j$ (each v is a linear combination of g_j's and in particular it suffices to define L on the g_j since they are the basis). Then $L_g(g_j) = gg_j$. Thus if we regard L_g as a matrix with respect to the basis G with the ordering $g_1, ..., g_n$, we have

$$(L_g)_{ij} = \begin{cases} 1, & g_i = gg_j; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & g = g_i g_j^{-1}; \\ 0, & \text{otherwise.} \end{cases}$$

This is because g sends g_j to g_i, so, to represent this in matrix notation, we think of g_j and g_i as basis vectors with zeros except in the jth or ith slot; then the matrix that sends g_j vector to g_i vector is precisely $(L_g)_{ij}$ as defined above.

In particular, when $i = j$, we get

$$(L_g)_{ii} = \begin{cases} 1, & g = 1; \\ 0, & g \neq 1. \end{cases}$$

From this it then follows that

$$\chi_L(g) = Tr(L_g) = \begin{cases} |G|, & g = 1; \\ 0, & g \neq 1. \end{cases}$$

(b) It follows from an earlier problem that

$$\chi_\phi = m_1\chi_{\phi_1} + \cdots + m_s\chi_{\phi_s}.$$

Then

$$\begin{aligned} \langle \chi_\phi, \chi_{\phi_i} \rangle &= \langle m_1\chi_{\phi_1} + \cdots + m_s\chi_{\phi_s}, \chi_{\phi_i} \rangle \\ &= m_1\langle \chi_{\phi_1}, \chi_{\phi_i} \rangle + \cdots + m_i\langle \chi_{\phi_i}, \chi_{\phi_i} \rangle + \cdots + m_s\langle \chi_{\phi_s}, \chi_{\phi_i} \rangle \\ &= m_1 \cdot 0 + \cdots + m_i \cdot 1 + \cdots + m_s \cdot 0 \\ &= m_i \end{aligned}$$

The next to last equality is the orthogonality relations for characters.

(c) Since the result from the previous part in fact shows that the decomposition of a representation is unique and that a representation is determined up to equivalence to its character, it suffices to check the inner product of χ_L with the χ_{ϕ_i}:

$$\langle \chi_L, \chi_{\phi_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_{\phi_i}(g)} = \frac{1}{|G|} |G| \cdot \overline{\chi_i(1)} = \chi_{\phi_i}(1) = \deg \phi_i = d_i.$$

Here we have used:

- Result about $\chi_L(g)$ from part (a);
- In class we observed that $\chi_{\phi_i}(1)$ is the trace of the identity matrix, and it hence gives the dimension of the vector space, i.e. the degree of the representation ϕ_i; and consequently
- $\chi_{\phi_i}(1)$ is a real number, and so $\overline{\chi_{\phi_i}(1)} = \chi_{\phi_i}(1)$.

(d) From the previous part, we know

$$L \sim d_1 \phi_1 \oplus \cdots \oplus d_s \phi_s.$$

Consequently,

$$\chi_L = d_1 \chi_{\phi_1} + \cdots + d_s \chi_{\phi_s}.$$

(we used this in part (b) already). Evaluating this equation at 1, we have

$$\chi_L(1) = d_1 \chi_{\phi_1}(1) + \cdots + d_s \chi_{\phi_s}(1).$$

We know from part (a) that $\chi_L(1) = |G|$ and we recalled in part (c) that $\chi_{\phi_i}(1) = d_i$. So the above equation becomes

$$|G| = d_1 \cdot d_1 + \cdots + d_s \cdot d_s,$$

as desired.