Math 306 Topics in Algebra, Spring 2013
Homework 8 Solutions

(1) (5 pts/part) Find the character tables of the following groups.
  (a) $\mathbb{Z}_3 \times \mathbb{Z}_3$
  (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Solution:

(a) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is abelian so each element is its own conjugacy class and each representation is one-dimensional. We can think of this group as generated by x and y where $x^3 = y^3 = 1$ and $xy = yx$. Thus a representation ϕ is determined by what it does on x and y. On each of those elements, ϕ has to behave like representation of $\mathbb{Z}_3$ and we know those are given by sending the generator to $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{4\pi i/3}$. We thus get the character table

	e	x	x^2	y	xy	x^2y	y^2	xy^2	x^2y^2
$\chi_{(0,0)} = \chi_{tr}$	1	1	1	1	1	1	1	1	1
$\chi_{(1,0)}$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$\chi_{(2,0)}$	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$\chi_{(0,1)}$	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$\chi_{(1,1)}$	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω
$\chi_{(2,1)}$	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1
$\chi_{(0,2)}$	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$\chi_{(1,2)}$	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$\chi_{(2,2)}$	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2

(b) This is essentially the same as in the previous part; each of the generators x, y, and z can be sent to either 1 or -1. This gives the table

	e	x	y	xy	z	xz	yz	xyz
$\chi_{(0,0,0)} = \chi_{tr}$	1	1	1	1	1	1	1	1
$\chi_{(1,0,0)}$	1	-1	1	-1	1	-1	1	-1
$\chi_{(0,1,0)}$	1	1	-1	-1	1	1	-1	-1
$\chi_{(1,1,0)}$	1	-1	-1	1	1	-1	-1	1
$\chi_{(0,0,1)}$	1	1	1	1	-1	-1	-1	-1
$\chi_{(1,0,1)}$	1	-1	1	-1	-1	1	-1	1
$\chi_{(0,1,1)}$	1	1	-1	-1	-1	-1	1	1
$\chi_{(1,1.1)}$	1	-1	-1	1	-1	1	1	-1

(2) (5 pts) Section 3.1, problem 41 (p. 89).

Solution: It is straightforward to show that N is indeed a subgroup: The product of commutators $x^{-1}y^{-1}xy$ and $a^{-1}b^{-1}ab$ is already in N since N is defined to be the subgroup *generated* by the commutators. The inverse of the commutator $x^{-1}y^{-1}xy$ is the commutator $(x^{-1}y^{-1}xy)^{-1} = y^{-1}x^{-1}yx$.

To see that the commutator subgroup is normal, we have, for all $x, y, g \in G$

$$gx^{-1}y^{-1}xyg^{-1} = gx(g^{-1}g)y(g^{-1}g)x^{-1}(g^{-1}g)y^{-1}g^{-1} = (gxg^{-1})^{-1}(gyg^{-1})^{-1}gxg^{-1}gyg^{-1} \in N.$$

To see that the quotient G/N is abelian, we have

$$(xN)(yN) = (yN)(xN) \iff (xy)N = (yx)N \iff (yx)^{-1}(xy) = x^{-1}y^{-1}xy \in N.$$

But by definition of N, $x^{-1}y^{-1}xy \in N$.

(3) (5 pts) Use the character table for D_4 we computed in class to determine the normal subgroups of D_4.

Solution: Taking all possible intersections of the kernels of the characters, we get that the normal subgroups are $\{1\}$, $\{1, \rho^2\}$, $\{1, \rho, \rho^2, \rho^3\}$, $\{1, \rho^2, \sigma, \sigma\rho^2\}$, $\{1, \rho^2, \sigma\rho, \sigma\rho^3\}$, D_4.

(4) (7 pts) Find the character table for A_4 and use it to find the normal subgroups of A_4.

Solution: Recall that

$$A_4 = \{1, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

This group has four conjugacy classes with representatives 1, $(12)(34)$, (123), and (132). There are thus four irreducible representations.

To find the 1-dimensional representations, it is not hard to see that the commutator subgroup is

$$[A_4, A_4] = \{1, (12)(34), (13)(24), (14)(23)\},$$

and so

$$A_4/[A_4, A_4] \cong \mathbb{Z}/3\mathbb{Z}.$$

(Since the commutator has order 4, the quotient must have order 3.)

So the three 1-dimensional representations of $\mathbb{Z}/3\mathbb{Z}$, given by $1 \mapsto \omega$ where $\omega^3 = 1$, lift to A_4.

We then have

$$12 = |A_4| = 1^2 + 1^2 + 1^2 + d^2,$$

from which it follows that the last representation has dimension 3. Denote its character by χ_4 and let $\chi_4((12)(34)) = a$, $\chi_4((123)) = b$, and $\chi_4((132)) = c$. We now use the orthogonality relations to obtain a system

$$\begin{aligned} 1 \cdot 3 \cdot 1 + 3 \cdot a \cdot 1 + 4 \cdot b \cdot 1 + 4 \cdot c \cdot 1 &= 0 \\ 1 \cdot 3 \cdot 1 + 3 \cdot a \cdot 1 + 4 \cdot b \cdot \omega + 4 \cdot c \cdot \omega^2 &= 0 \\ 1 \cdot 3 \cdot 1 + 3 \cdot a \cdot 1 + 4 \cdot b \cdot \omega^2 + 4 \cdot c \cdot \omega &= 0 \end{aligned}$$

from which it follows that $a = -1$, $b = c = 0$. The character table is thus

	1	$(12)(34)$	(123)	(132)
$\chi_1 = \chi_{tr}$	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

From the table we see that the only values of a character that equal the dimension of the corresponding representation is $\chi_2((12)(34)) = \chi_3((12)(34)) = 1$. Thus the only proper normal subgroup of A_4 is $\{1, (12)(34), (13)(24), (14)(23)\}$.

(5) (5 pts) Look up the definition of a (left) ideal of a ring R. Show that any (left) ideal of R is a (left) module over R.

Solution: If I is an ideal, the action $R \times I \to I$ is just multiplication in R, namely $(r, i) \mapsto ri$. Module axioms are satisfied because of the definition of an ideal (in particular, ri is always an element of I).

(6) (5 pts) Section 10.1, problem 9 (p. 344).

Solution: Let $\mathrm{Ann}(N)$ be the annihilator of N in R. Suppose $r \in \mathrm{Ann}(N)$, $a \in R$, and $n \in N$. We have to show $ar, ra \in \mathrm{Ann}(N)$. We have

$$(ar)(n) = a(rn) = 0$$

since $rn = 0$, and

$$(ra)(n) = r(an) = rn' = 0$$

for some $n' \in N$. Hence $\mathrm{Ann}(N)$ is a two-sided ideal of R.

(7) (5 pts) Section 10.1, problem 18 (p. 344).

Solution: Recall that an $F[x]$-module is a vector space V along with a linear transformation $T\colon V \to V$. It is then not hard to see that an $F[x]$ submodule is precisely a subspace which is T-stable, i.e. a subspace $W \subset V$ such that $T(W) \subset W$ (see page 341 for more details). If T is the rotation of $\mathbb{R}^2$ about the origin, the only subspaces that T-stable are $\{0\}$ and $\mathbb{R}^2$ (all other subspaces are lines through the origin, and these are rotated), so these are the only two possible submodules.

(8) (4 pts) Show that any ring with identity is a $\mathbb{Z}$-algebra.

Solution: By definition, a ring A with identity is a $\mathbb{Z}$-algebra if there is a ring homomorphism $f\colon \mathbb{Z} \to A$ sending unit to unit such that $f(\mathbb{Z}) \in Z(R)$. But sending $1 \in \mathbb{Z}$ to something determines the homomorphism. In other words, we can define f by $n \mapsto nf(1) = 1_R + 1_R + \cdot + 1_R$ (added n times; if $n < 0$, then we add -1_R to itself n times). It is easy to check that this is indeed a homomorphism.

Further,

$$m(nf(1)) = m(1_R + \cdot + 1_R) = m + m + \cdots + m = (1_R + \cdots + 1_R)m = (nf(1))m,$$

so $f(\mathbb{Z})$ is in the center of A.