Math 306, Spring 2012
First Midterm Exam

Name: ______________________________ Student ID: __________________________

Directions: Check that your test has 9 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (16 points) _____________
2. (5 points) _____________
3. (5 points) _____________
4. (5 points) _____________
5. (7 points) _____________
6. (8 points) _____________
7. (9 points) _____________

Total (out of 55): _____________

Curved score (out of 100): _____________

Letter grade: _____________

1. (2 pts each) Give examples of the following.

   (a) An integral domain which is not a field.

   (b) An infinite ring which is not an integral domain.

   (c) A unique factorization domain that is not a principal ideal domain.

   (d) A field of order 8.

   (e) A cubic polynomial in $\mathbb{Z}[x]$ that is irreducible by Eisenstein Criterion.

   (f) A quadratic polynomial in $\mathbb{Z}_5[x]$ that is irreducible.

   (g) A transcendental field extension $L\colon K$.

   (h) A field L such that $[L\colon \mathbb{Q}(\pi)] = 2$.

2. (5 pts) Let K be a subfield of $\mathbb{C}$. Let α be algebraic over K with minimal polynomial m. Show that there is an isomorphism

$$K[x]/(m) \cong K(\alpha).$$

Please be sure to verify that your isomorphism is well-defined. (Note that another way to do this is to exhibit a surjective homomorphism from $K[x]$ to $K(\alpha)$ whose kernel is (m).)

3. (5 pts) Determine whether the polynomial $f(x) = \frac{2}{9}x^5 + \frac{5}{3}x^4 + x^3 + \frac{1}{3}$ is reducible over $\mathbb{Q}$.

4. (5 pts) List all the irreducible monic quadratic polynomials in $\mathbb{Z}_2[x]$. Justify your answer.

5. (a) (4 pts) Show directly that the polynomial $f = x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$. Do not use general facts we know from class about polynomials of this form.

(b) (3 pts) Find $\alpha \in \mathbb{C}$ such that f is the minimal polynomial for α. (Hint: You should be able to just say what α is without any work.) What subgroup of $\mathbb{C}$ does α generate?

6. (4 pts each) Let p and q be distinct primes in $\mathbb{Z}$.

   (a) Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is a simple extension of $\mathbb{Q}$.

   (b) Prove that $\alpha = \sqrt{p} + \sqrt{q}$ is algebraic over $\mathbb{Q}$ by exhibiting (with justification) an appropriate quartic polynomial in $\mathbb{Q}[x]$ with α as a root.

7. Let α and β be complex numbers, and let $\mathbb{A}$ be the collection of algebraic numbers over $\mathbb{Q}$. Do not assume in this problem that $\mathbb{A}$ is a field.

(a) (4 pts) Prove that, if α and β belong to $\mathbb{A}$, then $\alpha + \beta$ and $\alpha\beta$ both belong to $\mathbb{A}$.

(b) (5 pts) If $\beta \in \mathbb{A}$ is nonzero, let $f = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + x^n$ be its minimum polynomial over $\mathbb{Q}$. Prove that $\beta^{-1} \in \mathbb{A}$ by explicitly finding a monic polynomial in $\mathbb{Q}[x]$ which has β^{-1} as a root.