Math 306, Spring 2012
Second Midterm Exam, April 12, 2012

Name: ______________________________ Student ID: __________________________

Directions: Check that your test has 8 pages, including this one and the blank one on the bottom (which you can use as scratch paper or to continue writing out a solution if you run out room elsewhere). Please **show all your work**. **Write neatly: solutions deemed illegible will not be graded, so no credit will be given.** This exam is closed book, closed notes. You have 70 minutes. Good luck!

1. (16 points) ____________
2. (15 points) ____________
3. (7 points) ____________
4. (9 points) ____________
5. (12 points) ____________
6. (9 points) ____________

Total (out of 68): ____________

Curved score (out of 100): ____________

Letter grade: ____________

1. (2 pts each) Give examples of the following.

   (a) An angle that cannot be trisected.

   (b) A splitting field of $x^4 + 4 \in \mathbb{Q}[x]$.

   (c) An irreducible cubic polynomial $f \in \mathbb{Q}[x]$ such that $[L\colon \mathbb{Q}] = 6$, where L is the splitting field of f.

   (d) A nonnormal extension.

   (e) An inseparable extension.

   (f) A nontrivial $\mathbb{Q}$-automorphism $\sigma\colon \mathbb{Q}(\sqrt{2}, \sqrt{3}) \to \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

   (g) An extension $L\colon K$ such that $\mathrm{Gal}(L/K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (h) An extension $L\colon K$ such that $\mathrm{Gal}(L/K) \cong \mathbb{Z}_3$.

2. (a) (10 pts) Prove that, if L is the splitting field of some $f \in K[x]$, then the extension $L\colon K$ is normal. (This is one direction of Theorem 5.)

(b) (5 pts) Prove that, if L is algebraic over K and every element of L belongs to an intermediate field that is normal over K, then L is normal over K.

3. (7 pts) Let p be prime and let $b \in \mathbb{Z}_p$ be fixed. If α is a root of the polynomial $f = x^p - x + b \in \mathbb{Z}_p[x]$, prove that $\mathbb{Z}_p(\alpha)$ is a normal extension of $\mathbb{Z}_p$. (Hint: You may use without proof that, for all $c \in \mathbb{Z}_p$, $c^p = c$.)

4. (a) (4 pts) State a lemma relating the degrees of $K(\alpha, \beta)$, $K(\alpha)$, and $K(\beta)$ over K if the degrees of the minimal polynomials of α and β are relatively prime.

(b) (5 pts) Use part (a) to show that the degree of the splitting field of $x^p - 2 \in \mathbb{Q}[x]$, p prime, is $p(p-1)$.

5. (a) (5 pts) Let $\zeta = e^{2\pi i/5}$. Prove that $\mathbb{Q}(\zeta)$ is a normal extension of $\mathbb{Q}$ and determine with proof the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. (Hint: It suffices to verify the order of one particular element in the Galois group.)

(b) (7 pts) Let $n \in \mathbb{Z}_{\geq 3}$ be fixed and let $\zeta = e^{2\pi i/n}$. Prove that $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is abelian.

6. (a) (4 pts) State the Galois Correspondence. Be sure to define the maps $*$ and $\dagger$.

(a) (5 pts) Show by example that maps $*$ and $\dagger$ need not be inverses of each other in general.